MA121-001

Tuesday, December 4

Chapter 6: (today)

6.1, 6.2:

final exam:

THURSDAY, DECEMBER 13
8:00 – 11:00 am SAS 2203

EVAL:

80% → 1 pt. bonus
85% → 2 pt. bonus
90% → 3 pt. bonus
95% → 4 pt. bonus
100% → 5 pt bonus

Chapter 6:

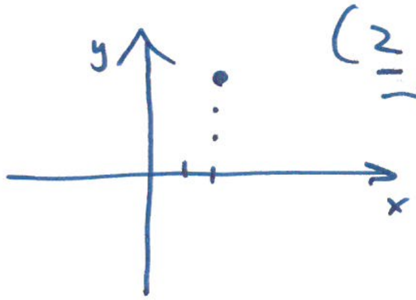
FUNCTIONS OF MORE THAN ONE VARIABLE

<u>before today</u>: (SINGLE VARIABLE)

- $f(x) = x^2 - 6x + 11$

$f(2) = (2)^2 - 6(2) + 11 = 4 - 12 + 11 = \underline{3}$

<u>2D</u>



$(2, 3)$ ordered pair

(x, y)

<u>today</u>: (2 VARIABLES)

$f(x, y) = x^2 + 4xy - 11y^2 + 5y - 2x + 4$

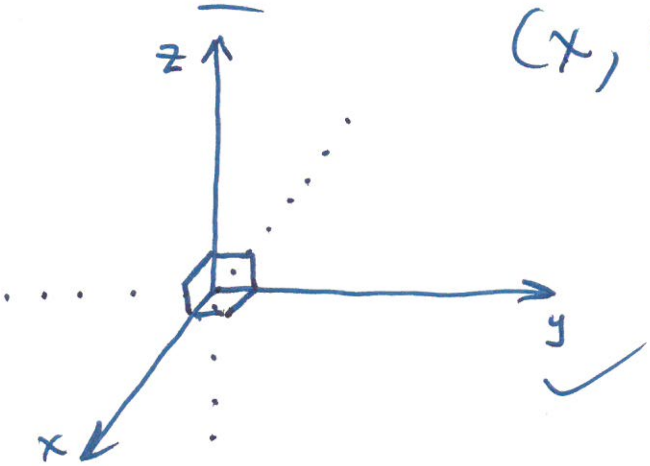
$f(2, 1) = (2)^2 + 4(2)(1) - 11(1)^2 + 5(1) - 2(2) + 4 =$

$f(2, 1) = 4 + 8 - 11 + 5 - 4 + 4 = 6$

$(2, 1, 6)$ ordered triple

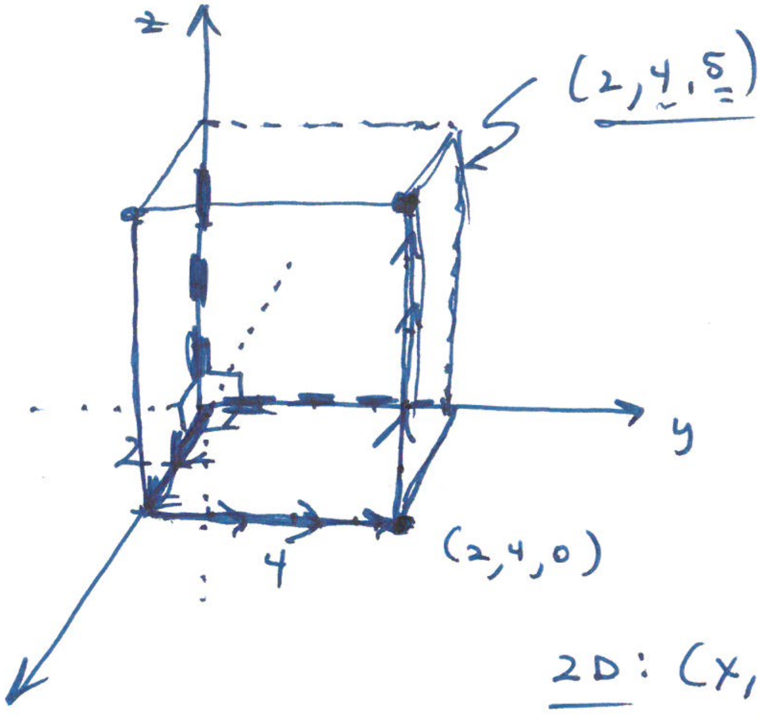
(x, y, z)

<u>3D</u>

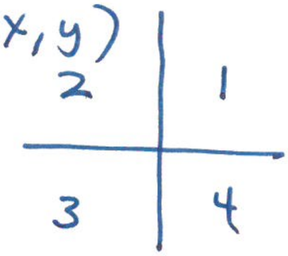


mutually $\perp$ axes

(2, 4, 5)



2D: (x, y)



QUADRANTS

3D:
(x, y, z)

(+, +, +)
(+, −, +)
(−, +, +)
(+, +, −)
⋮
8 OCTANTS

wind chill:

$$f(T,v) = 35.74 + 0.6215(T) - 35.75(v^{.16}) + .4275(T)(v^{.16})$$

T = air temp. (F) T = 32°
v = wind speed (mph) v = 10 mph

$$f(32,10) = 35.74 + 0.6215(32) - 35.75(10)^{.16} + .4275(32)(10)^{.16}$$

$$f(32,10) = \underline{23.72°}$$

(32°, 10mph, 23.72°)

6.2: $f(x,y) = $ ______

first deriv: $f'(x)$

* 1st order partial deriv:

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

* y's FIXED
temporarily (treat y-values as CONSTANTS)

$$f(x,y) = x^2 - 8xy + 4y^2 - 8y + 5x + 2$$

$$f_x = 2x - 8y(1) + 0 - 0 + 5 + 0 = 2x - 8y + 5$$

x's are variables
(y's are constants)

$$f_x = 2x - 8y + 5 \quad ✓$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_y = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

y's → VARIABLES
(x's → FIXED)
↑ treat temp. as CONSTANTS

$$f(x,y) = x^2 - 8xy + 4y^2 - 8y + 5x + 2$$

$$f_y = 0 - 8x(1) + 8y - 8 + 0 + 0$$

$$f_y = -8x + 8y - 8 \quad ✓$$

y's – VARIABLES
(x's – CONSTANTS)

1st order partial deriv:

1. f_x
2. f_y

2^{nd} order partial deriv:

1. f_{xx}
2. f_{xy}
3. f_{yy}
4. f_{yx}

(2) = (4)

$f(x,y) = x^2 \cdot y^3 + 4xy + 4y^2$

find f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{yx}:

$\Rightarrow f_x = y^3 \cdot (2x) + 4y(1) + 0 = 2xy^3 + 4y$

$f_x = 2xy^3 + 4y$

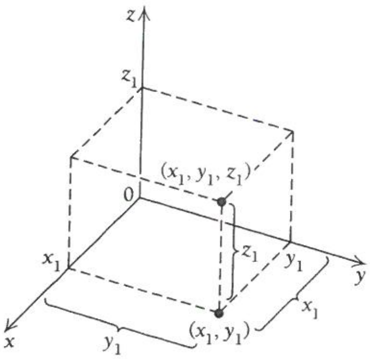
1. $f_{xx} = 2y^3(1) + 0$

$f_{xx} = 2y^3$

2. $f_{xy} = 2x(3y^2) + 4 = 6xy^2 + 4 = f_{xy}$

$\Rightarrow f_y = x^2(3 \cdot y^2) + 4x(1) + 8y = 3x^2y^2 + 4x + 8y$

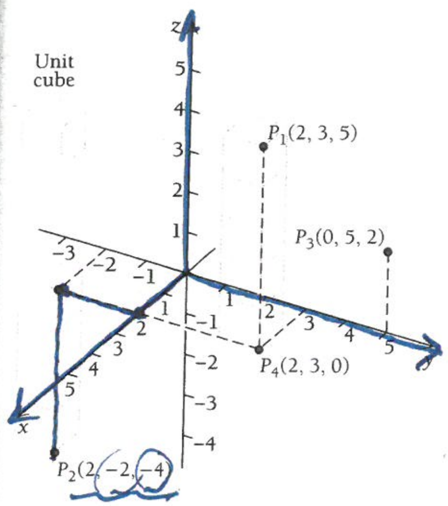
1. $f_{yy} = 3x^2(2y) + 0 + 8 = 6x^2y + 8$
2. $f_{yx} = 3y^2(2x) + 4 + 0 = 6xy^2 + 4 = f_{yx}$



Geometric Interpretations

Visually, a function of two variables, $z = f(x, y)$, can be thought of as matching a point (x_1, y_1) in the xy-plane with the number z_1 on a number line. Thus, to graph a function of two variables, we need a three-dimensional coordinate system. The axes are generally placed as shown to the left. The line z, called the z-axis, is perpendicular to the xy-plane at the origin.

To help visualize this, think of looking into the corner of a room, where the floor is the xy-plane and the z-axis is the intersection of the two walls. To plot a point (x_1, y_1, z_1), we locate the point (x_1, y_1) in the xy-plane and move up or down in space according to the value of z_1.

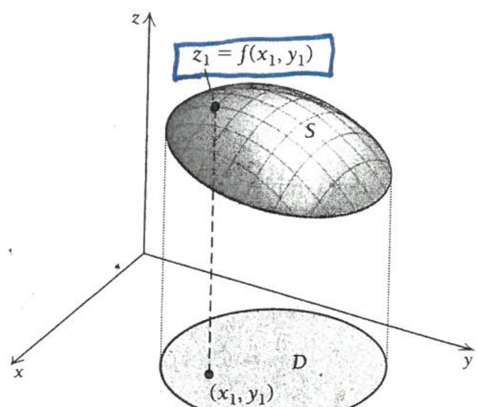


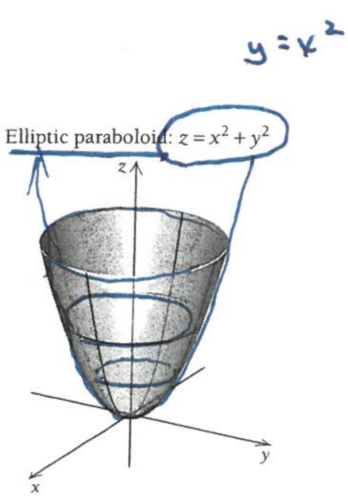
EXAMPLE 7 Plot these points:

$P_1(2, 3, 5),$
$P_2(2, -2, -4),$
$P_3(0, 5, 2),$
and $P_4(2, 3, 0).$

Solution The solution is shown at the left. ■

The *graph* of a function of two variables, $z = f(x, y)$, consists of ordered triples (x_1, y_1, z_1), where $z_1 = f(x_1, y_1)$. This graph takes the form of a **surface**. The **domain** of such a function is the set of all points in the xy-plane for which f is defined.





Elliptic paraboloid: $z = x^2 + y^2$

EXAMPLE 8 Find the domain of each two-variable function.

a) $f(x, y) = x^2 + y^2$

b) $g(x, y) = \sqrt{1 - x^2 - y^2}$

c) $h(x, y) = x^2 + y^2 + \dfrac{1}{x^2 + y^2}$

Solution

a) Since we can square any real number and add any two squares, f is defined for all x and all y. Therefore, the domain of f is

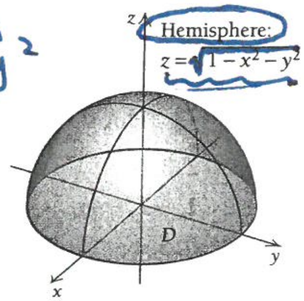
$$D = \{(x, y) \mid -\infty < x < \infty, \quad -\infty < y < \infty\}.$$

The graph of f is a surface called an *elliptic paraboloid*. A satellite dish is an example of an elliptic paraboloid: the weak incoming signals bounce off the interior surface of the paraboloid and collect at a single point, called the *focus*, thus amplifying the signal.

b) For $g(x, y)$ to exist, we must have $1 - x^2 - y^2 \geq 0$, or $x^2 + y^2 \leq 1$. The domain of g is

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

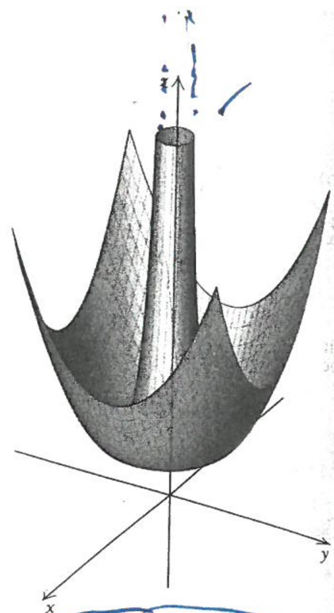
The graph of g is a surface called a *hemisphere,* of radius 1. Its domain is a filled-in circle of radius 1 on the xy-plane. We can think of the domain of g as the "shadow" it casts on the xy-plane.



c) Since zero cannot be in the denominator, we must have $x^2 + y^2 \neq 0$. Therefore, x and y cannot be 0 simultaneously. The domain of h is

$$D = \{(x, y) \mid (x, y) \neq (0, 0)\}.$$

The graph of h is shown at right. **6 ☑**



$z = x^2 + y^2 + \dfrac{1}{x^2 + y^2}$

Quick Check 6 ☑

Find the domain of each two-variable function.

a) $f(x, y) = \dfrac{x + y}{x - y}$

b) $g(x, y) = \dfrac{1}{x - 2} + \dfrac{2}{3 + y}$

c) $h(x, y) = \ln(y - x^3)$

TECHNOLOGY CONNECTION

Exploratory

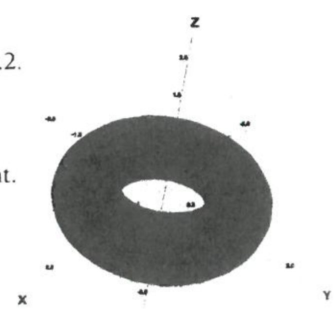
A useful and inexpensive app is Quick Graph, a graphing calculator that creates visually appealing 3D graphs of functions of two variables. It has full graphing interactivity, with touch-based zoom and scroll features.

Some functions and their graphs are presented here.

EXAMPLE 1 Graph:
$(1 - \sqrt{x^2 + y^2})^2 + z^2 = 0.2.$
This is entered as follows:

```
(1-sqrt(x^2+y^2))^2+z^2=0.2
```

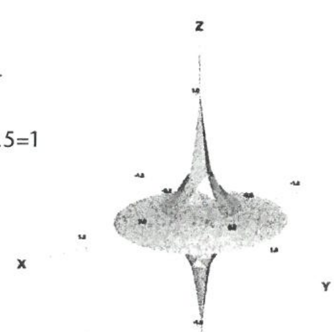
The graph is shown at the right.



EXAMPLE 2 Graph:
$|(2x^2 + 2y^2)^{0.25}| + \sqrt{|z|} = 1.$
This is entered as follows:

```
abs((2x^2+2y^2)^0.25)+(abs(z))^0.5=1
```

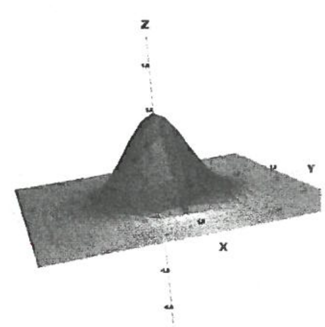
The graph is shown at the right.



EXAMPLE 3 Graph: $z = e^{-4(x^2+y^2)}$.
This is entered as follows:

```
z=e^(-4(x^2+y^2))
```

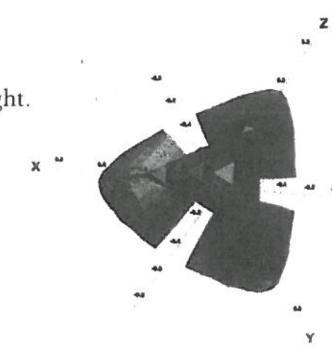
The graph is shown at the right.



EXAMPLE 4 Graph: $(xy)^2 + (yz)^2 + (zx)^2 = xyz.$
This is entered as follows:

```
(xy)^2+(yz)^2+(xz)^2=xyz
```

The graph is shown at the right.



(continued)

6.2

- Find the partial derivatives of a given function.
- Evaluate partial derivatives.
- Find the four second-order partial derivatives of a function in two variables.

Teaching Tip

You may need to remind students at times to treat variables as constants. For example, the derivative of $4y^2$, with respect to x, is zero because the derivative of any constant is zero.

Partial Derivatives

Finding Partial Derivatives

Consider the function f given by

$$z = f(x, y) = x^2y^3 + xy + 4y^2.$$

Suppose we fix y at 3. Then

$$f(x, 3) = x^2(3^3) + x(3) + 4(3^2) = 27x^2 + 3x + 36.$$

Note that we now have a function of only one variable. Taking the first derivative with respect to x, we have

$$54x + 3.$$

In general, without replacing y with a specific number, we can consider y fixed. Then f becomes a function of x alone, and we can calculate its derivative with respect to x. This is called the *partial derivative of f with respect to x*, denoted by

$$\frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial z}{\partial x}.$$

Now, let's again consider the function

$$z = f(x, y) = x^2y^3 + xy + 4y^2.$$

The color blue indicates the variable x when we fix y and treat it as a constant. The expressions y^3, y, and y^2 are then also treated as constants. We have

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2y^3 + xy + 4y^2)\\ &= 2xy^3 + (1)y + 0\\ &= 2xy^3 + y.\end{aligned}$$

Similarly, we find $\partial f/\partial y$ or $\partial z/\partial y$ by fixing x (treating it as a constant) and calculating the derivative with respect to y. From

$$z = f(x, y) = x^2y^3 + xy + 4y^2,$$ The color blue indicates the variable.

we get

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2y^3 + xy + 4y^2)\\ &= x^2(3y^2) + x(1) + 8y\\ &= 3x^2y^2 + x + 8y.\end{aligned}$$

A definition of partial derivatives is as follows.

DEFINITION

For $z = f(x, y)$, the **partial derivatives with respect to x and y** are

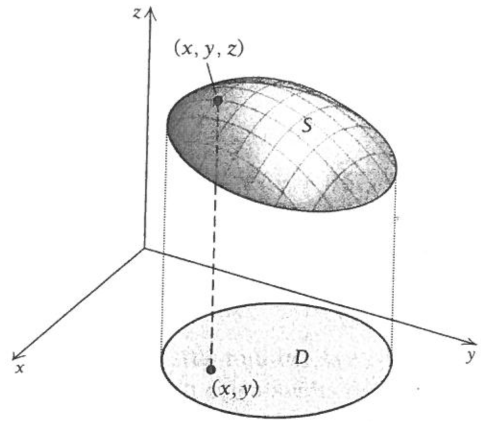
$$\frac{\partial z}{\partial x} = \lim_{h\to 0}\frac{f(x + h, y) - f(x, y)}{h} \quad \text{and} \quad \frac{\partial z}{\partial y} = \lim_{h\to 0}\frac{f(x, y + h) - f(x, y)}{h}.$$

We can find partial derivatives of functions of any number of variables. Since the earlier theorems for finding derivatives apply, we rarely need to use the definition to find a partial derivative.

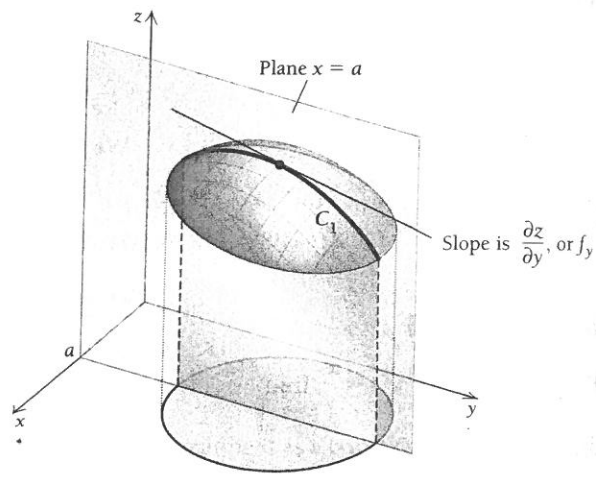
The Geometric Interpretation of Partial Derivatives

The roof of this building is a smooth continuous surface. The slope at a point on the surface depends on the direction in which the tangent line is oriented.

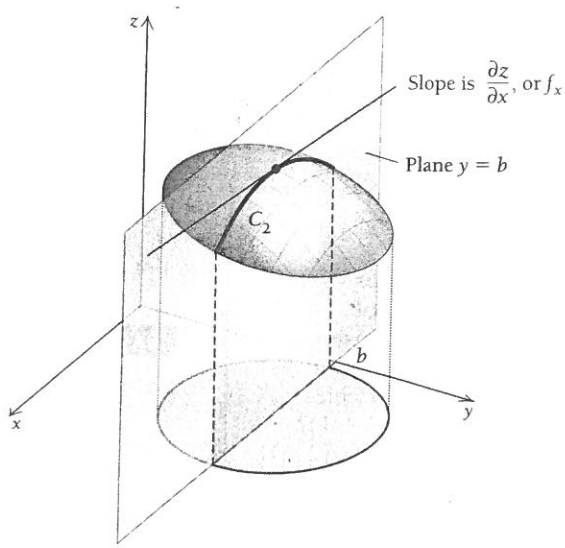
The graph of a function of two variables $z = f(x, y)$ is a surface S, which might have a graph similar to the one shown to the right, where each input pair (x, y) in the domain D has only one output, $z = f(x, y)$.



Now suppose we hold x fixed at the value a. The set of all points for which $x = a$ is a plane parallel to the yz-plane; thus, when x is fixed at a, y and z vary along that plane, as shown to the right. The plane $x = a$ in the figure cuts the surface along the curve C_1. The partial derivative f_y gives the slope of tangent lines to this curve, in the positive y-direction.



Similarly, if we hold y fixed at the value b, we obtain a curve C_2, as shown to the right. The partial derivative f_x gives the slope of tangent lines to this curve, in the positive x-direction.



Higher-Order Partial Derivatives

Consider

$$z = f(x, y) = 3xy^2 + 2xy + x^2.$$

Then $$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = 3y^2 + 2y + 2x.$$

Suppose we continue and find the first partial derivative of $\partial z/\partial x$ with respect to y. This will be a **second-order partial derivative** of the original function z, denoted by

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3y^2 + 2y + 2x) = 6y + 2.$$

The notation $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$ is often expressed as

$$\frac{\partial^2 z}{\partial y\, \partial x} \quad \text{or} \quad \frac{\partial^2 f}{\partial y\, \partial x}.$$

We can also denote the preceding partial derivative using the notation f_{xy}:

$$f_{xy} = 6y + 2.$$

Teaching Tip

You might ask students how many second-order partial derivatives are possible for functions with two variables before discussing the four results and their notation.

Note that in the notation f_{xy}, x and y are in the order (left to right) in which the differentiation is done, but in

$$\frac{\partial^2 f}{\partial y\, \partial x},$$

the order of x and y is reversed. In each case, the differentiation with respect to x is done first, followed by differentiation with respect to y.

Notation for the four second-order partial derivatives is as follows.

DEFINITION **Second-Order Partial Derivatives**

1. $\frac{\partial^2 z}{\partial x\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial x} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}$ Take the partial derivative with respect to x, and then with respect to x again.
2. $\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial y\, \partial x} = f_{xy}$ Take the partial derivative with respect to x, and then with respect to y.
3. $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial x\, \partial y} = f_{yx}$ Take the partial derivative with respect to y, and then with respect to x.
4. $\frac{\partial^2 z}{\partial y\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial y} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$ Take the partial derivative with respect to y, and then with respect to y again.

EXAMPLE 5 For

$$z = f(x, y) = x^2y^3 + x^4y + xe^y,$$

find the four second-order partial derivatives.

Solution

a) $\frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}(2xy^3 + 4x^3y + e^y)$ Differentiate f with respect to x.

$= 2y^3 + 12x^2y$ Differentiate f_x with respect to x.