MA 121-002

Tuesday, December 4

Chapter 6: (today)

6.1, 6.2:

final exam:

{ THURSDAY, DECEMBER 13
1:00 – 4:00 pm SAS 2203 }

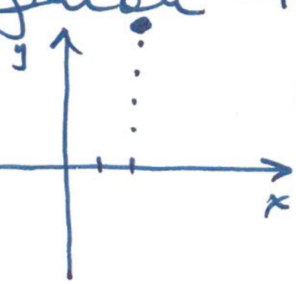
EVAL:

80% → 1 pt bonus
85% → 2 pt bonus
90% → 3 pt bonus
95% → 4 pt bonus
100% → 5 pt bonus

6.1: FUNCTIONS OF MORE THAN ONE VARIABLE

prior to today: (SINGLE VARIABLE)

2D:



$f(x) = x^2 - 5x + 11$

$f(2) = 2^2 - 5(2) + 11 = 5$

$4 - 10 + 11$

ordered pair

$(2, 5)$

today:

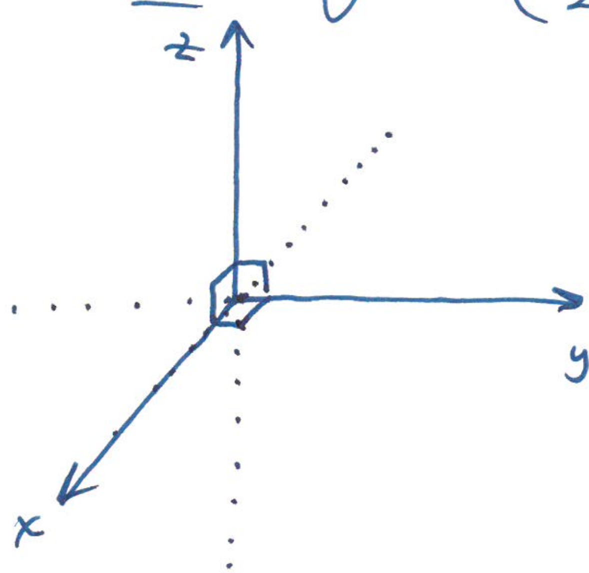
$$f(x,y) = x^2 + 4xy - 5y^2 + 4x - 3y + 1$$

$$f(2,1) = (2)^2 + 4(2)(1) - 5(1)^2 + 4(2) - 3(1) + 1$$

$$f(2,1) = 4 + 8 - 5 + 8 - 3 + 1$$

$$f(2,1) = 13$$

3D



$(2, 1, 13)$ ← $f(2,1)$

ordered triple

(x, y, z)

mutually $\perp$

2D:

2	1
3	4

OCTANTS

3D: $(+,+,+)$

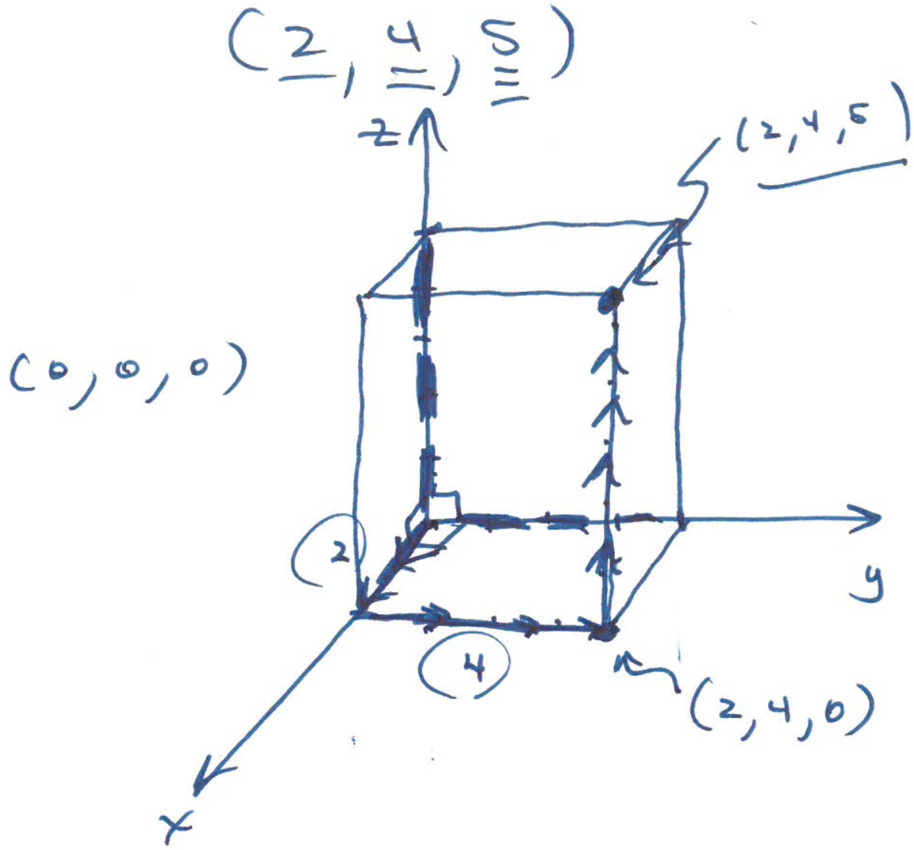
$(+,+,-)$

$(+,-,+)$

$(-,+,+)$

⋮

8



$f(x, y) = $ ___

wind chill:

$$f(T, V) = 35.74 + .6215\,(T^1) - 35.75\,(V^{.16}) + .4275\,(T^1)\cdot(V^{.16})$$

T = temp (F)

V = wind speed (mph)

$T = 32^\circ$

$V = 10$ mph

$$f(32^\circ, 10 \text{ mph}) = 35.74 + .6215(32) - 35.75\,(10)^{.16} + .4275(32)(10)^{.16}$$

$$= 23.72^\circ$$

$(32, 10, 23.72)$

heat index:

DERIVATIVES:

1^{st} DERIV & 2^{nd} DERIV

1^{st} order partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

partial deriv with respect to $\underline{\underline{\underline{x}}}$

(x IS VARIABLE)

y IS FIXED

(y is tempor. treated as a CONSTANT)

$$f(x,y) = 4x^2 + 8xy - 11y^2 + 4x - 14$$

$$f_{\underline{x}} = 8x + 8y(1) - 0 + 4 - 0$$

$$\boxed{f_x = 8x + 8y + 4}$$

$$f_x(8, -5) =$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_y = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

y's are VARIABLE

x's are FIXED

(x's are tempor treated as constants)

$f(x,y) = \underbrace{4x^2} + \underbrace{8xy} - \underbrace{11y^2} + \underbrace{4x} \underbrace{-14}$

$f_y = 0 + 8x(1) \cancel{4} - 22y + 0 - 0$

$$\begin{cases} f_y = 8x - 22y \\ f_x = 8x + 8y + 4 \end{cases}$$

$d(e^{5x}) = e^{5x} \cdot 5$

$f(x,y) = \underbrace{e^{xy}} + \underbrace{y \cdot \ln x}$ $\quad \underbrace{e^{x\cdot 5}}$ $\leftarrow y$

$\rightarrow f_x = (e^{xy}) \cdot (y) + y \cdot \left(\frac{1}{x}\right)$

$\rightarrow f_y = (e^{xy}) \cdot (x) + (\ln x) \cdot (1)$

2nd order partial deriv: (4)

f_{xx} ; f_{xy} ; f_{yy} ; f_{yx}

=

$f(x,y) = \underbrace{3x^2} \underbrace{-2xy} + \underbrace{4y}$

$f_x = 6x - 2y(1) + 0 = \boxed{\underbrace{6x} \underbrace{-2y} = f_x}$

2nd order partial deriv:

(a) $f_{xx} = 6 - 0 = \boxed{6 = f_{xx}}$

(b) $f_{xy} = 0 - 2 = \boxed{-2 = f_{xy}}$

$f_y = 0 - 2x + 4 = \boxed{-2x + 4 = f_y}$ =

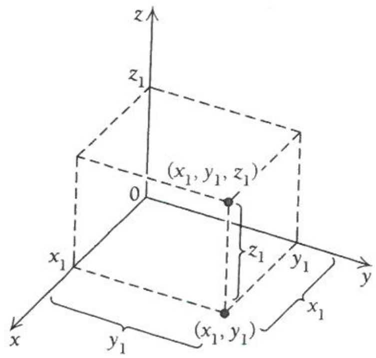
(a) $f_{yy} = 0 + 0 = \boxed{0 = f_{yy}}$

(b) $f_{yx} = -2 + 0 = \boxed{-2 = f_{yx}}$

exam:

$f(x,y) =$ ______

find $f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{yx}$

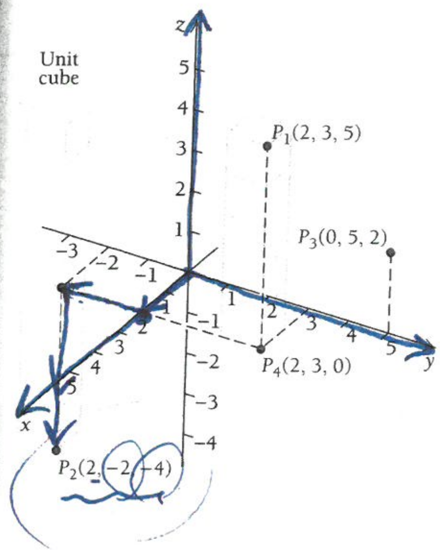


Geometric Interpretations

Visually, a function of two variables, $z = f(x, y)$, can be thought of as matching a point (x_1, y_1) in the xy-plane with the number z_1 on a number line. Thus, to graph a function of two variables, we need a three-dimensional coordinate system. The axes are generally placed as shown to the left. The line z, called the z-axis, is perpendicular to the xy-plane at the origin.

To help visualize this, think of looking into the corner of a room, where the floor is the xy-plane and the z-axis is the intersection of the two walls. To plot a point (x_1, y_1, z_1), we locate the point (x_1, y_1) in the xy-plane and move up or down in space according to the value of z_1.

Unit cube



EXAMPLE 7 Plot these points:

$P_1(2, 3, 5)$,

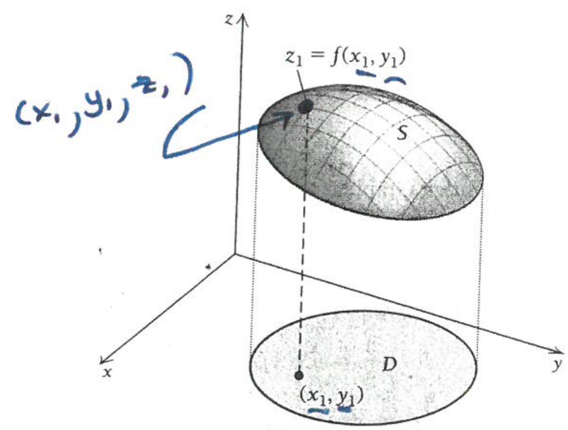
$P_2(2, -2, -4)$,

$P_3(0, 5, 2)$,

and $P_4(2, 3, 0)$.

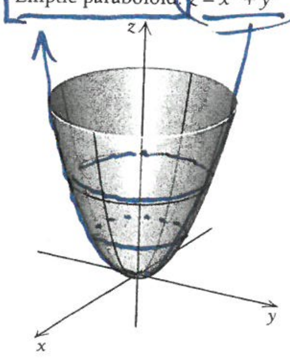
Solution The solution is shown at the left. ■

The *graph* of a function of two variables, $z = f(x, y)$, consists of ordered triples (x_1, y_1, z_1), where $z_1 = f(x_1, y_1)$. This graph takes the form of a **surface**. The **domain** of such a function is the set of all points in the xy-plane for which f is defined.



$y = x^2$

Elliptic paraboloid: $z = x^2 + y^2$



EXAMPLE 8 Find the domain of each two-variable function.

a) $f(x, y) = x^2 + y^2$

b) $g(x, y) = \sqrt{1 - x^2 - y^2}$

c) $h(x, y) = x^2 + y^2 + \dfrac{1}{x^2 + y^2}$

Solution

a) Since we can square any real number and add any two squares, f is defined for all x and all y. Therefore, the domain of f is

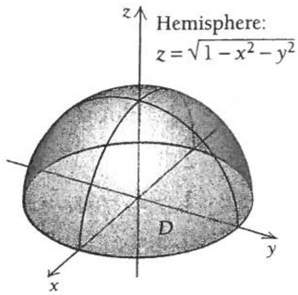
$$D = \{(x, y) \mid -\infty < x < \infty, \quad -\infty < y < \infty\}.$$

The graph of f is a surface called an *elliptic paraboloid*. A satellite dish is an example of an elliptic paraboloid: the weak incoming signals bounce off the interior surface of the paraboloid and collect at a single point, called the *focus*, thus amplifying the signal.

b) For $g(x, y)$ to exist, we must have $1 - x^2 - y^2 \geq 0$, or $x^2 + y^2 \leq 1$. The domain of g is

$$D = \{(x, y) | x^2 + y^2 \leq 1\}.$$

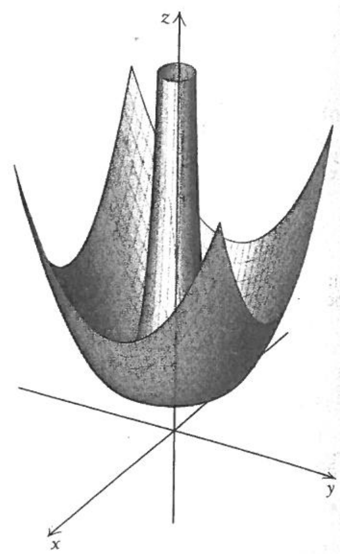
The graph of g is a surface called a *hemisphere,* of radius 1. Its domain is a filled-in circle of radius 1 on the xy-plane. We can think of the domain of g as the "shadow" it casts on the xy-plane.



c) Since zero cannot be in the denominator, we must have $x^2 + y^2 \neq 0$. Therefore, x and y cannot be 0 simultaneously. The domain of h is

$$D = \{(x, y) | (x, y) \neq (0, 0)\}.$$

The graph of h is shown at right. **6** ☑



$z = x^2 + y^2 + \dfrac{1}{x^2 + y^2}$

Quick Check 6 ☑

Find the domain of each two-variable function.

a) $f(x, y) = \dfrac{x + y}{x - y}$

b) $g(x, y) = \dfrac{1}{x - 2} + \dfrac{2}{3 + y}$

c) $h(x, y) = \ln(y - x^3)$

TECHNOLOGY CONNECTION

Exploratory

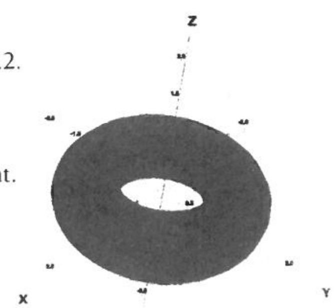
A useful and inexpensive app is Quick Graph, a graphing calculator that creates visually appealing 3D graphs of functions of two variables. It has full graphing interactivity, with touch-based zoom and scroll features.

Some functions and their graphs are presented here.

EXAMPLE 1 Graph:
$(1 - \sqrt{x^2 + y^2})^2 + z^2 = 0.2.$
This is entered as follows:

(1-sqrt(x^2+y^2))^2+z^2=0.2

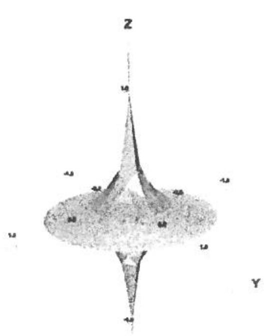
The graph is shown at the right.



EXAMPLE 2 Graph:
$|(2x^2 + 2y^2)^{0.25}| + \sqrt{|z|} = 1.$
This is entered as follows:

abs((2x^2+2y^2)^0.25)+(abs(z))^0.5=1

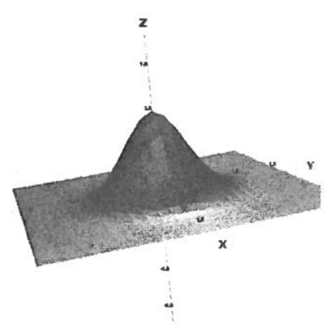
The graph is shown at the right.



EXAMPLE 3 Graph: $z = e^{-4(x^2+y^2)}$.
This is entered as follows:

z=e^(-4(x^2+y^2))

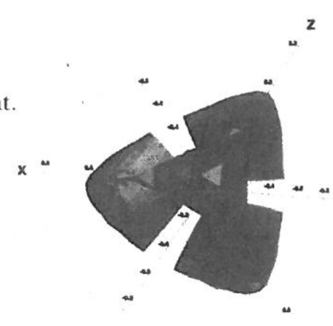
The graph is shown at the right.



EXAMPLE 4 Graph: $(xy)^2 + (yz)^2 + (zx)^2 = xyz.$
This is entered as follows:

(xy)^2+(yz)^2+(xz)^2=xyz

The graph is shown at the right.



(continued)

6.2 Partial Derivatives

- Find the partial derivatives of a given function.
- Evaluate partial derivatives.
- Find the four second-order partial derivatives of a function in two variables.

Teaching Tip

You may need to remind students at times to treat variables as constants. For example, the derivative of $4y^2$, with respect to x, is zero because the derivative of any constant is zero.

Finding Partial Derivatives

Consider the function f given by

$$z = f(x, y) = x^2y^3 + xy + 4y^2.$$

Suppose we fix y at 3. Then

$$f(x, 3) = x^2(3^3) + x(3) + 4(3^2) = 27x^2 + 3x + 36.$$

Note that we now have a function of only one variable. Taking the first derivative with respect to x, we have

$$54x + 3.$$

In general, without replacing y with a specific number, we can consider y fixed. Then f becomes a function of x alone, and we can calculate its derivative with respect to x. This is called the *partial derivative of f with respect to x*, denoted by

$$\frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial z}{\partial x}.$$

Now, let's again consider the function

$$z = f(x, y) = x^2y^3 + xy + 4y^2.$$

The color blue indicates the variable x when we fix y and treat it as a constant. The expressions y^3, y, and y^2 are then also treated as constants. We have

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2y^3 + xy + 4y^2)\\ &= 2xy^3 + (1)y + 0\\ &= 2xy^3 + y.\end{aligned}$$

Similarly, we find $\partial f/\partial y$ or $\partial z/\partial y$ by fixing x (treating it as a constant) and calculating the derivative with respect to y. From

$$z = f(x, y) = x^2y^3 + xy + 4y^2, \qquad \text{The color blue indicates the variable.}$$

we get

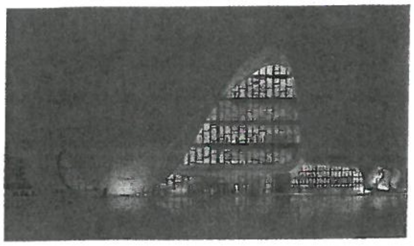
$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2y^3 + xy + 4y^2)\\ &= x^2(3y^2) + x(1) + 8y\\ &= 3x^2y^2 + x + 8y.\end{aligned}$$

A definition of partial derivatives is as follows.

> **DEFINITION**
>
> For $z = f(x, y)$, the **partial derivatives with respect to x and y** are
>
> $$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \quad \text{and} \quad \frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

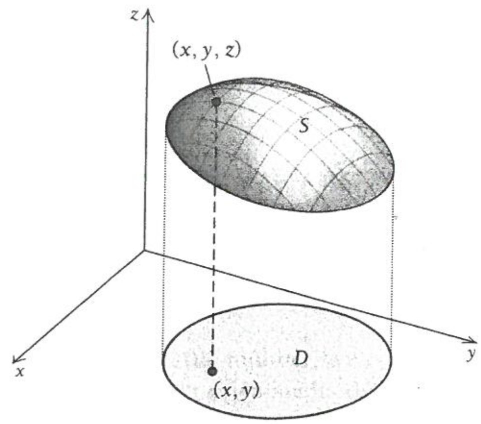
We can find partial derivatives of functions of any number of variables. Since the earlier theorems for finding derivatives apply, we rarely need to use the definition to find a partial derivative.

The Geometric Interpretation of Partial Derivatives

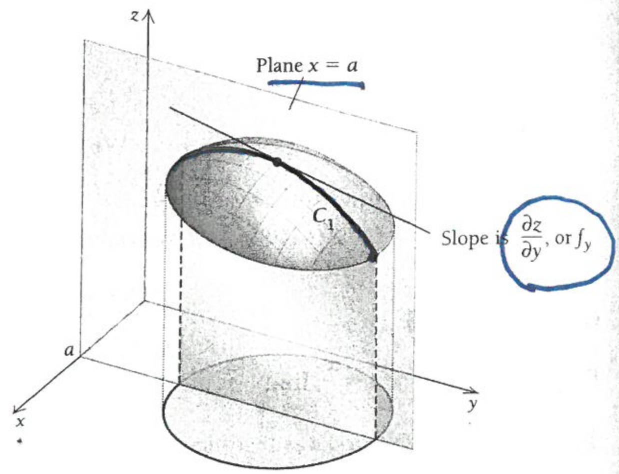


The roof of this building is a smooth continuous surface. The slope at a point on the surface depends on the direction in which the tangent line is oriented.

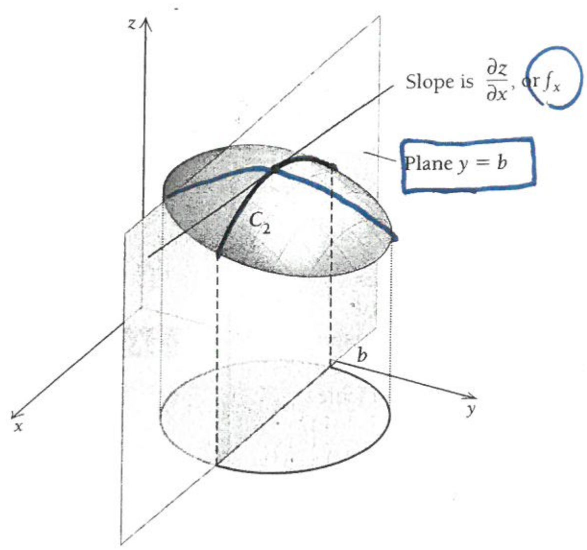
The graph of a function of two variables $z = f(x, y)$ is a surface S, which might have a graph similar to the one shown to the right, where each input pair (x, y) in the domain D has only one output, $z = f(x, y)$.



Now suppose we hold x fixed at the value a. The set of all points for which $x = a$ is a plane parallel to the yz-plane; thus, when x is fixed at a, y and z vary along that plane, as shown to the right. The plane $x = a$ in the figure cuts the surface along the curve C_1. The partial derivative f_y gives the slope of tangent lines to this curve, in the positive y-direction.



Similarly, if we hold y fixed at the value b, we obtain a curve C_2, as shown to the right. The partial derivative f_x gives the slope of tangent lines to this curve, in the positive x-direction.



Higher-Order Partial Derivatives

Consider

$$z = f(x, y) = 3xy^2 + 2xy + x^2.$$

Then $\dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial x} = 3y^2 + 2y + 2x.$

Suppose we continue and find the first partial derivative of $\partial z/\partial x$ with respect to y. This will be a **second-order partial derivative** of the original function z, denoted by

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3y^2 + 2y + 2x) = 6y + 2.$$

The notation $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$ is often expressed as

$$\frac{\partial^2 z}{\partial y\, \partial x} \quad \text{or} \quad \frac{\partial^2 f}{\partial y\, \partial x}.$$

We can also denote the preceding partial derivative using the notation f_{xy}:

$$f_{xy} = 6y + 2.$$

Teaching Tip

You might ask students how many second-order partial derivatives are possible for functions with two variables before discussing the four results and their notation.

Note that in the notation f_{xy}, x and y are in the order (left to right) in which the differentiation is done, but in

$$\frac{\partial^2 f}{\partial y\, \partial x},$$

the order of x and y is reversed. In each case, the differentiation with respect to x is done first, followed by differentiation with respect to y.

Notation for the four second-order partial derivatives is as follows.

DEFINITION Second-Order Partial Derivatives

1. $\dfrac{\partial^2 z}{\partial x\, \partial x} = \dfrac{\partial^2 f}{\partial x\, \partial x} = \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 f}{\partial x^2} = f_{xx}$ Take the partial derivative with respect to x, and then with respect to x again.
2. $\dfrac{\partial^2 z}{\partial y\, \partial x} = \dfrac{\partial^2 f}{\partial y\, \partial x} = f_{xy}$ Take the partial derivative with respect to x, and then with respect to y.
3. $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial^2 f}{\partial x\, \partial y} = f_{yx}$ Take the partial derivative with respect to y, and then with respect to x.
4. $\dfrac{\partial^2 z}{\partial y\, \partial y} = \dfrac{\partial^2 f}{\partial y\, \partial y} = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 f}{\partial y^2} = f_{yy}$ Take the partial derivative with respect to y, and then with respect to y again.

EXAMPLE 5 For

$$z = f(x, y) = x^2y^3 + x^4y + xe^y,$$

find the four second-order partial derivatives.

Solution

a) $\dfrac{\partial^2 f}{\partial x^2} = f_{xx} = \dfrac{\partial}{\partial x}(2xy^3 + 4x^3y + e^y)$ Differentiate f with respect to x.

$= 2y^3 + 12x^2y$ Differentiate f_x with respect to x.