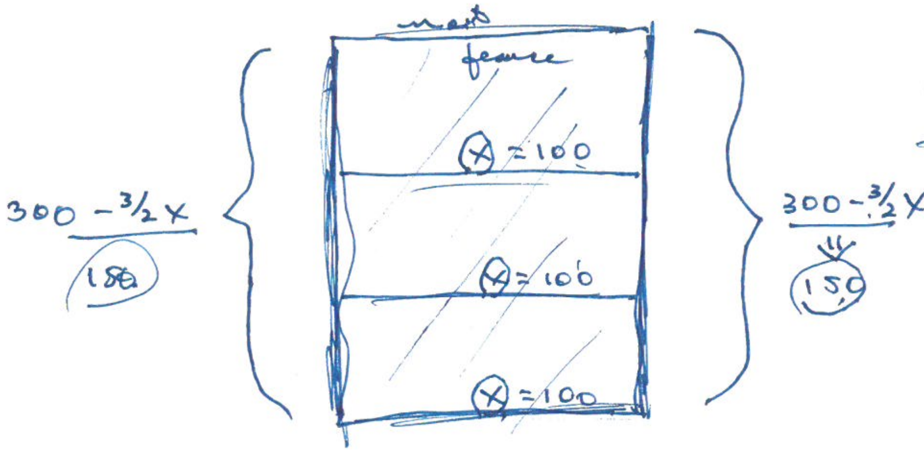
MA 121-003
Monday, October 15

2.5:

MAX/MIN WORD PROBLEMS
(OPTIMIZATION PROBLEMS)



600 ft.

$$\frac{600 - 3x}{2} = 300 - \frac{3}{2}x$$

$$(300 - 3/2x) + (300 - 3/2x) + (x) + (x) + (x)$$

MAX AREA

$$A = (x)\cdot(300 - 3/2x)$$

$$A(x) = 300x - 3/2x^2$$

$$A'(x) = 300 - 3/2(2\cdot x)$$

$$A'(x) = \boxed{300 - 3x = 0}$$

$$\frac{300}{3} = \frac{3x}{3} \qquad x = 100'$$

$$A''(x) = -3$$

∴ CONC. DOWN
∴ MAX.

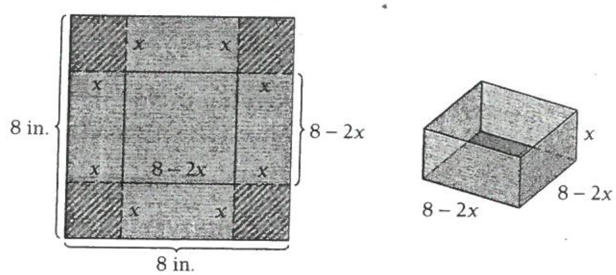
$$A(100) = (100)\left(300 - \tfrac{3}{2}(100)\right)$$
$$= (100)(150)$$
$$= 15{,}000 \text{ ft}^2$$

A Strategy for Solving Maximum–Minimum Problems

1. Read the problem carefully. If relevant, make a drawing.
2. Make a list of appropriate variables and constants, noting what varies, what stays fixed, and what units are used. Label the measurements on your drawing, if one exists.
3. Translate the problem to an equation involving a quantity Q to be maximized or minimized. Try to represent Q in terms of the variables of step 2.
4. Try to express Q as a function of one variable. Use the procedures developed in Sections 2.1–2.4 to determine the maximum or minimum values and the points at which they occur.

EXAMPLE 2 **Maximizing Volume.** From a thin piece of cardboard 8 in. by 8 in., square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

Solution We make a drawing in which x is the length, in inches, of each square to be cut. It is important to note that since the original square is 8 in. by 8 in., after the smaller squares are removed, the lengths of the sides of the box will be $(8 - 2x)$ in. by $(8 - 2x)$ in.



After the four small squares are removed and the sides are folded up, the volume V of the resulting box is

$$V = l \cdot w \cdot h = (8 - 2x) \cdot (8 - 2x) \cdot x,$$

Section Summary

- In many real-life applications, we need to determine the minimum or maximum value of a function modeling a situation.
- Maximum profit occurs at those x-values for which $R'(x) = C'(x)$ and $R''(x) < C''(x)$.

2.5 Exercise Set

1. Of all numbers whose sum is 70, find the two that have the maximum product. That is, maximize $Q = xy$, where $x + y = 70$. Maximum $Q = 1225$; $x = 35$, $y = 35$

2. Of all numbers whose sum is 50, find the two that have the maximum product. That is, maximize $Q = xy$, where $x + y = 50$. Maximum $Q = 625$; $x = 25$, $y = 25$

3. Of all numbers whose difference is 6, find the two that have the minimum product. Minimum product $= -9$; $x = 3$, $y = -3$

4. Of all numbers whose difference is 4, find the two that have the minimum product. Minimum product $= -4$; $x = 2$, $y = -2$

5. Maximize $Q = xy^2$, where x and y are positive numbers such that $x + y^2 = 4$. Maximum $Q = 4$; $x = 2$, $y = \sqrt{2}$, or approximately 1.414

6. Maximize $Q = xy^2$, where x and y are positive numbers such that $x + y^2 = 1$. Maximum $Q = \frac{1}{4}$; $x = \frac{1}{2}$, $y = \sqrt{\frac{1}{2}}$

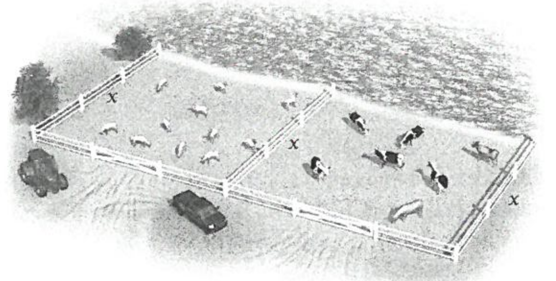
7. Minimize $Q = x^2 + 2y^2$, where $x + y = 3$. Minimum $Q = 6$; $x = 2$, $y = 1$

8. Minimize $Q = 2x^2 + 3y^2$, where $x + y = 5$. Minimum $Q = 30$; $x = 3$, $y = 2$

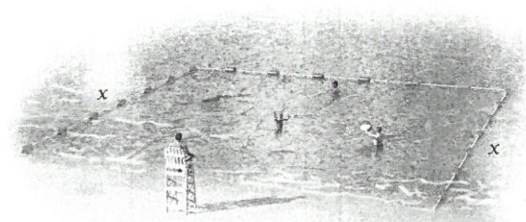
9. Maximize $Q = xy$, where x and y are positive numbers such that $x + \frac{4}{3}y^2 = 1$. Maximum $Q = \frac{1}{3}$; $x = \frac{2}{3}$, $y = \frac{1}{2}$

10. Maximize $Q = xy$, where x and y are positive numbers such that $\frac{4}{3}x^2 + y = 16$. Maximum $Q = 21\frac{1}{3}$; $x = 2$, $y = 10\frac{2}{3}$

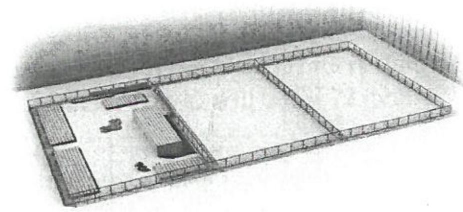
11. **Maximizing area.** A rancher wants to enclose two rectangular areas near a river, one for sheep and one for cattle. There are 240 yd of fencing available. What is the largest total area that can be enclosed? Maximum area $= 4800$ yd^2 ($x = 40$ yd, $y = 120$ yd)



12. **Maximizing area.** A lifeguard needs to rope off a rectangular swimming area in front of Long Lake Beach, using 180 yd of rope and floats. What dimensions of the rectangle will maximize the area? What is the maximum area? (Note that the shoreline is one side of the rectangle.) Maximum area $= 4050$ yd^2; width is 45 yd, and length (parallel to shoreline) is 90 yd



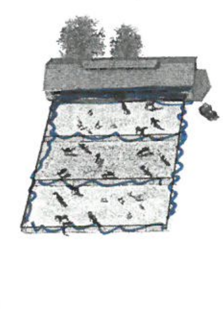
13. **Maximizing area.** Hentz Industries plans to enclose three parallel rectangular areas for sorting returned goods. The three areas are within one large rectangular area and 1200 yd of fencing is available. What is the largest total area that can be enclosed?



Maximum area $= 45{,}000$ yd^2; (150 yd by 300 yd)

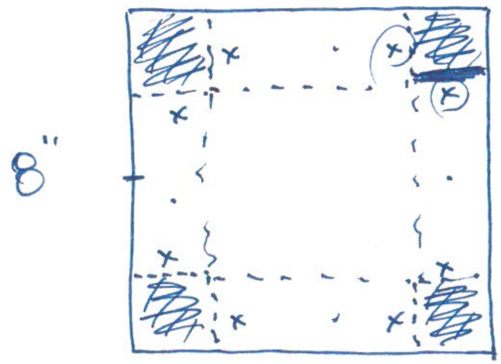
14. **Maximizing area.** Grayson Farms plans to enclose three parallel rectangular livestock pens within one large rectangular area using 600 ft of fencing. One side of the enclosure is a pre-existing stone wall.

    a) If the three rectangular pens have their longer sides parallel to the stone wall, find the largest possible total area that can be enclosed. 15,000 ft^2

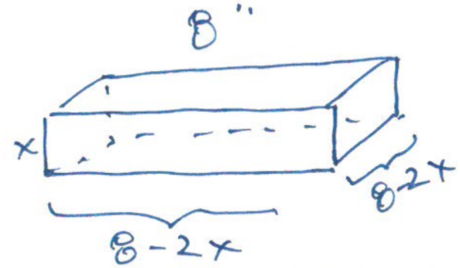


endpts:

$0 < x < 4$



MAX VOL: $\cancel{}$



$8-2(4/3)$

$\frac{24}{3} - 8/3$

$$\boxed{V = x(8-2x)(8-2x)}$$

$$V(x) = x(64 - 32x + 4x^2)$$

$$V(x) = 4x^3 - 32x^2 + 64x$$

$$V'(x) = 12x^2 - 64x + 64 = 0$$

$$4(3x^2 - 16x + 16) = 0$$

$$4(3x-4)(x-4) = 0$$

$3x - 4 = 0$

$3x = 4$

$x = 4/3$

$x - 4 \neq 0$

$\cancel{x = 4}$

if $x = 4/3$

DIM: $4/3" \cdot 16/3" \cdot 16/3"$

MAX VOL: $\frac{1024}{27}$ in^3

VALIDATE WITH THE SECOND DERIV:

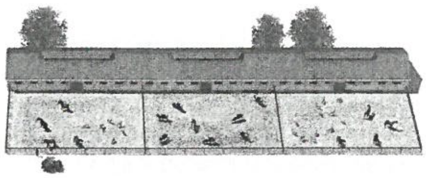
$$V''(x) = 24x - 64$$

$$V''(4/3) = 24(4/3) - 64 = -$$

$\therefore$ CONC DOWN

$\therefore$ MAX

b) If the three rectangular pens have their shorter sides perpendicular to the stone wall, find the largest possible total area that can be enclosed. 22,500 ft^2



15. Maximizing area. Of all rectangles that have a perimeter of 42 ft, find the dimensions of the one with the largest area. What is its area? Maximum area $= 110.25\ \text{ft}^2$; $x = 10.5$ ft; $y = 10.5$ ft

16. Maximizing area. A carpenter is building a rectangular shed with a fixed perimeter of 54 ft. What are the dimensions of the largest shed that can be built? What is its area? $x = 13.5$ ft, $y = 13.5$ ft; maximum area $= 182.25\ \text{ft}^2$

17. Maximizing volume. From a thin piece of cardboard 20 in. by 20 in., square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume? Dimensions: $13\frac{1}{3}$ in. by $13\frac{1}{3}$ in. by $3\frac{1}{3}$ in.; maximum volume $= 592\frac{16}{27}\ \text{in}^3$

18. Maximizing volume. From a 50-cm-by-50-cm sheet of aluminum, square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume? Dimensions; $33\frac{1}{3}$ cm by $33\frac{1}{3}$ cm by $8\frac{1}{3}$ cm; maximum volume $= 9259\frac{7}{27}\ \text{cm}^3$

19. Minimizing surface area. Mendoza Soup Company is constructing an open-top, square-based, rectangular metal tank that will have a volume of $32\ \text{ft}^3$. What dimensions will minimize surface area? What is the minimum surface area? Dimensions: 4 ft by 4 ft by 2 ft; minimum surface area $= 48\ \text{ft}^2$

20. Minimizing surface area. Drum Tight Containers is designing an open-top, square-based, rectangular box that will have a volume of $62.5\ \text{in}^3$. What dimensions will minimize surface area? What is the minimum surface area? Dimensions: 5 in. by 5 in. by 2.5 in.; minimum surface area $= 75\ \text{in}^2$

21. Minimizing surface area. Open Air Waste Management is designing a rectangular construction dumpster that will be twice as long as it is wide and must hold $12\ \text{yd}^3$ of debris. Find the dimensions of the dumpster that will minimize its surface area. 2.08 yd by 4.16 yd by 1.387 yd

22. Minimizing surface area. Ever Green Gardening is designing a rectangular compost container that will be twice as tall as it is wide and must hold $18\ \text{ft}^3$ of composted food scraps. Find the dimensions of the compost container with minimal surface area (include the bottom and top). 1.89 ft by 2.52 ft by 3.78 ft

APPLICATIONS

Business and Economics

Maximizing profit. *For Exercises 23–28, find the maximum profit and the number of units that must be produced and sold in order to yield the maximum profit. Assume that revenue, $R(x)$, and cost, $C(x)$, are in dollars for Exercises 23–26.*

23. $R(x) = 50x - 0.5x^2, \quad C(x) = 4x + 10$ \$1048; 46 units

24. $R(x) = 50x - 0.5x^2, \quad C(x) = 10x + 3$ \$797; 40 units

25. $R(x) = 2x, \quad C(x) = 0.01x^2 + 0.6x + 30$ \$19; 70 units

26. $R(x) = 5x, \quad C(x) = 0.001x^2 + 1.2x + 60$ \$3550; 1900 units

27. $R(x) = 9x - 2x^2, \quad C(x) = x^3 - 3x^2 + 4x + 1$; assume that $R(x)$ and $C(x)$ are in thousands of dollars, and x is in thousands of units. \$5481; 1667 units

28. $R(x) = 100x - x^2, C(x) = \frac{1}{3}x^3 - 6x^2 + 89x + 100$; assume that $R(x)$ and $C(x)$ are in thousands of dollars, and x is in thousands of units. \$182,333; 11,000 units

29. Maximizing profit. Riverside Appliances is marketing a new refrigerator. It determines that in order to sell x refrigerators, the price per refrigerator must be

$$p = 280 - 0.4x.$$

It also determines that the total cost of producing x refrigerators is given by

$$C(x) = 5000 + 0.6x^2.$$

a) Find the total revenue, $R(x)$. $R(x) = x(280 - 0.4x)$
b) Find the total profit, $P(x)$. $P(x) = -x^2 + 280x - 5000$
c) How many refrigerators must the company produce and sell in order to maximize profit? 140 refrigerators
d) What is the maximum profit? \$14,600
e) What price per refrigerator must be charged in order to maximize profit? \$224/refrigerator

30. Maximizing profit. Raggs, Ltd., a clothing firm, determines that in order to sell x suits, the price per suit must be

$$p = 150 - 0.5x.$$

It also determines that the total cost of producing x suits is given by (b) $P(x) = -0.75x^2 + 150x - 4000$

$$C(x) = 4000 + 0.25x^2.$$

a) Find the total revenue, $R(x)$. $R(x) = x(150 - 0.5x)$
b) Find the total profit, $P(x)$.
c) How many suits must the company produce and sell in order to maximize profit? 100 suits
d) What is the maximum profit? \$3500
e) What price per suit must be charged in order to maximize profit? \$100/suit

31. Maximizing profit. Gritz-Charlston is a 300-unit luxury hotel. All rooms are occupied when the hotel charges \$80 per day for a room. For every increase of x dollars in the daily room rate, there are x rooms vacant. Each occupied room costs \$22 per day to service and maintain. What should the hotel charge per day in order to maximize profit? \$201 per day

32. Maximizing revenue. Edwards University wants to determine what price to charge for tickets to football games. At a price of \$18 per ticket, attendance averages 40,000 people per game. Every decrease of \$3 to the ticket price adds 10,000 people to the average attendance. Every person at a game spends an average of \$4.50 on concessions. What price per ticket should be charged to maximize revenue? How many people will attend at that price? \$12.75/ticket; 57,500 people

33. Maximizing parking tickets. Oak Glen currently employs 8 patrol officers who each write an average of 24 parking tickets per day. For every additional officer placed on patrol, the average number of parking tickets per day written by each officer decreases by 4. How many additional officers should be placed on patrol in order to maximize the number of parking tickets written per day? 1 fewer officer

34. Maximizing yield. Hood Apple Farm yields an average of 30 bushels of apples per tree when 20 trees are planted on an acre of ground. If 1 more tree is planted per acre, the yield decreases by 1 bushel (bu) per tree as a result of crowding. How many trees should be planted on an acre in order to get the highest yield? 25 trees/acre

35. Nitrogen prices. During 2001, nitrogen prices fell by 41%. Over the same year, nitrogen demand went up by 12%. (*Source*: *Chemical Week*.)

a) Assuming a linear change in demand, find the demand function, $q(x)$, by finding the equation of the line that passes through the points $(1, 1)$ and $(0.59, 1.12)$. Here x is the price as a fraction of the January 2001 price, and $q(x)$ is the demand as a fraction of the demand in January. $q(x) = -\frac{12}{41}x + \frac{53}{41}$;

b) As a percentage of the January 2001 price, what should the price of nitrogen be to maximize revenue? 221% of the January 2001 price

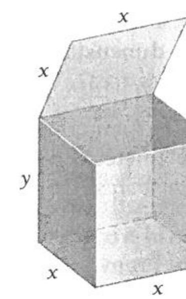
36. Vanity license plates. According to a pricing model, increasing the fee for vanity license plates by \$1 decreases the percentage of a state's population that will request them by 0.04%. (*Source*: E. D. Craft, "The demand for vanity (plates): Elasticities, net revenue maximization, and deadweight loss," *Contemporary Economic Policy*, Vol. 20, 133–144 (2002).)

a) Recently, the fee for vanity license plates in Maryland was \$25, and the percentage of the state's population that had vanity plates was 2.13%. Use this information to construct the demand function, $q(x)$, for the percentage of Maryland's population that will request vanity license plates for a fee of x dollars. $q(x) = 3.13 - 0.04x$

b) Find the fee, x, that will maximize revenue from vanity plates. \$39.13

37. Maximizing revenue. When the Marchant Theater charges \$5 for admission, there is an average attendance of 180 people. For every \$0.10 increase in admission, there is a loss of 1 customer from the average number. What admission should be charged in order to maximize revenue? \$11.50

38. Minimizing costs. A rectangular box with a volume of 320 ft^3 is to be constructed with a square base and top. The cost per square foot for the bottom is 15¢, for the top is 10¢, and for the sides is 2.5¢. What dimensions will minimize the cost? 4 ft by 4 ft by 20 ft



39. Minimizing cost. A rectangular parking area measuring 5000 ft^2 is to be enclosed on three sides using chain-link fencing that costs \$4.50 per foot. The fourth side will be a wooden fence that costs \$7 per foot. What dimensions will minimize the total cost to enclose this area, and what is the minimum cost (rounded to the nearest dollar)? 62.55 ft by 79.93 ft; \$1439

40. Minimizing cost. A rectangular garden measuring 1200 yd^2 is to be enclosed on two parallel sides by stone wall that costs \$35 per yd and the other two sides by wooden fencing that costs \$28 per yd. What dimensions will minimize the total cost of enclosing this garden, and what is the minimum cost (rounded to the nearest dollar)? 30.98 yd by 38.73 yd; \$4337

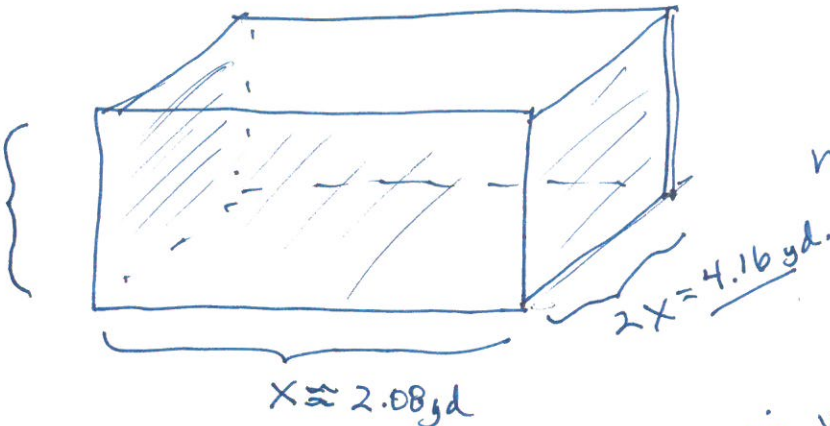
41. Maximizing area. Bradley Publishing decides that each page in a new book must have an area of 73.125 in^2, a 0.75-in. margin at the top and at the bottom of each page, and a 0.5-in. margin on each of the sides. What should the outside dimensions of each page be so that the printed area is a maximum? 6.98 in. by 10.47 in.

42. Minimizing inventory costs. A sporting goods store sells 100 pool tables per year. It costs \$20 to store one pool table for a year. To reorder, there is a fixed cost of \$40 per shipment plus \$16 for each pool table. How many times per year should the store order pool tables, and in what lot size, in order to minimize inventory costs? Order 5 times/yr; lot size is 20.

43. Minimizing inventory costs. A pro shop in a bowling center sells 200 bowling balls per year. It costs \$4 to store one bowling ball for a year. To reorder, there is a fixed cost of \$1, plus \$0.50 for each bowling ball. How many times per year should the shop order bowling balls, and in what lot size, in order to minimize inventory costs? Order 20 times/yr; lot size is 10.

$\frac{6}{2.08^2} \xrightarrow{} 1.387$ yd.

$\frac{6}{x^2} = H??$



MIN. SURFACE AREA:

is V = 12 ??

$(\cancel{x})(2\cancel{x})\left(\frac{6}{\cancel{x^2}}\right) \stackrel{?}{=} \underline{12}$

$V = 12$

$V = x \cdot 2x \cdot H$

$12 = x \cdot 2x \cdot H$

solve for H

$\frac{12}{2x^2} = \frac{\cancel{2x^2} \cdot H}{\cancel{2x^2}} = \frac{6}{x^2}$

surface area:

$2\left(x \cdot \frac{6}{x^2}\right) + 2\left(2x \cdot \frac{6}{x^2}\right) + (x \cdot 2x) = S$

$\frac{12}{x} + \frac{24}{x} + 2x^2 = S(x)$

$36x^{-1} \rightarrow \frac{36}{x} + 2x^2 = S(x)$

$-36x^{-2} + 4x = 0$

$\frac{-36}{x^2} + 4x = 0$

$\frac{4x}{1} = \frac{36}{x^2}$

$36 = 4x^3$

$9 = x^3$

$x = \sqrt[3]{9} \approx 2.08$ yd.

$S''(x) = 72x^{-3} + 4$

$S''(2.08) = \frac{72}{(2.08)^3} + 4 = +$

$\therefore$ CONC. UP

$\therefore$ MIN

44. **Minimizing inventory costs.** A retail outlet for Boxowitz Calculators sells 720 calculators per year. It costs \$2 to store one calculator for a year. To reorder, there is a fixed cost of \$5, plus \$2.50 for each calculator. How many times per year should the store order calculators, and in what lot size, in order to minimize inventory costs? Order 12 times/yr; lot size is 60.

45. **Minimizing inventory costs.** Bon Temps Surf and Scuba Shop sells 360 surfboards per year. It costs \$8 to store one surfboard for a year. Each reorder costs \$10, plus an additional \$5 for each surfboard ordered. How many times per year should the store order surfboards, and in what lot size, in order to minimize inventory costs? Order 12 times/yr; lot size is 30.

46. **Minimizing inventory costs.** Repeat Exercise 44 using the same data, but assume yearly sales of 256 calculators with the fixed cost of each reorder set at \$4. Order 8 times/yr; lot size is 32.

47. **Minimizing inventory costs.** Repeat Exercise 45 using the same data, but change the reorder costs from an additional \$5 per surfboard to \$6 per surfboard. Order 12 times/yr; lot size is 30.

48. **Minimizing surface area.** A closed-top cylindrical container is to have a volume of 250 in^2. What dimensions (radius and height) will minimize the surface area? $r \approx 3.414$ in., $h \approx 6.828$ in.

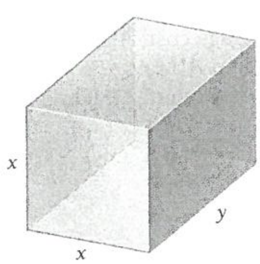
49. **Minimizing surface area.** An open-top cylindrical container is to have a volume of 400 cm^2. What dimensions (radius and height) will minimize the surface area? $r \approx 5.03$ cm, $h \approx 5.03$ cm

50. **Minimizing cost.** Assume that the costs of the materials for making the cylindrical container described in Exercise 48 are \$0.005/in^2 for the circular base and top and \$0.003/in^2 for the wall. What dimensions will minimize the cost of materials? $r \approx 2.879$ in., $h \approx 9.598$ in.

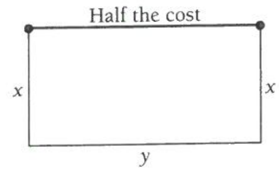
51. **Minimizing cost.** Assume that the costs of the materials for making the cylindrical container described in Exercise 49 are \$0.0015/cm^2 for the base and \$0.008/cm^2 for the wall. What dimensions will minimize the cost of materials? $r \approx 4.08$ cm, $h \approx 7.65$ cm

General Interest

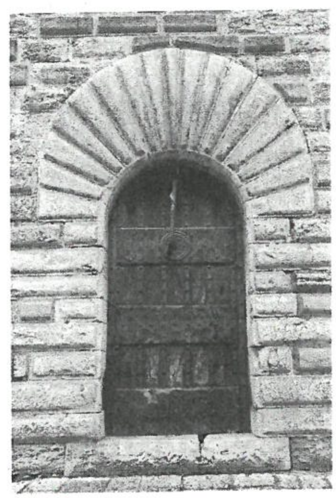
52. **Maximizing volume.** The postal service places a limit of 84 in. on the combined length and girth of (distance around) a package to be sent parcel post. What dimensions of a rectangular box with square cross-section will contain the largest volume that can be mailed? (*Hint:* There are two different girths.) 14 in. by 14 in. by 28 in.

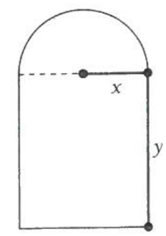


53. **Minimizing cost.** A rectangular play area of 48 yd^2 is to be fenced off in a person's yard. The next-door neighbor agrees to pay half the cost of the fence on the side of the play area that lies along the property line. What dimensions will minimize the cost of the fence? 6 yd by 8 yd (8-yd side is opposite the side shared with neighbor)



54. **Maximizing light.** A Norman window is a rectangle with a semicircle on top. Suppose that the perimeter of a particular Norman window is to be 24 ft. What should its dimensions be in order to allow the maximum amount of light to enter through the window? $x \approx 3.36$ ft, $y \approx 3.36$ ft





55. **Maximizing light.** Repeat Exercise 54, but assume that the semicircle is to be stained glass, which transmits only half as much light as clear glass does. $x \approx 2.76$ ft, $y \approx 4.92$ ft

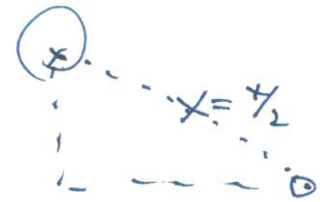
56–110. Use a spreadsheet to numerically verify the result of Exercises 1–55. Left to the student

SYNTHESIS

111. For what positive number is the sum of its reciprocal and five times its square a minimum? $\sqrt[3]{0.1}$, or approximately 0.4642

112. For what positive number is the sum of its reciprocal and four times its square a minimum? 0.5

113. **Business: maximizing profit.** The amount of money that customers deposit in a bank in savings accounts is directly proportional to the interest rate that the bank pays on that money. Suppose a bank is able to loan out all the money deposited in its savings accounts at an interest rate of 18%. What interest rate should it pay on its savings accounts in order to maximize profit? 9%



45.) MIN. INVENTORY COST: lot size: x

on site — reorder cost

$$\boxed{\text{COST} = (8^{00})\left(\frac{x}{2}\right) + (5\cdot x + 10^{00})\left(\frac{360}{x}\right)}$$

ASSUME — # of times to reorder

$$\underset{\uparrow}{N}\cdot\underset{\uparrow}{x} = 360 \qquad N = ? = \frac{360}{x}$$

$$C(x) = 4x + 1800 + \frac{3600}{x}$$