MA 121 - 003

Monday, December 3

Chapter 6: (today)

6.1, 6.2:

final exam: MONDAY, DECEMBER 10

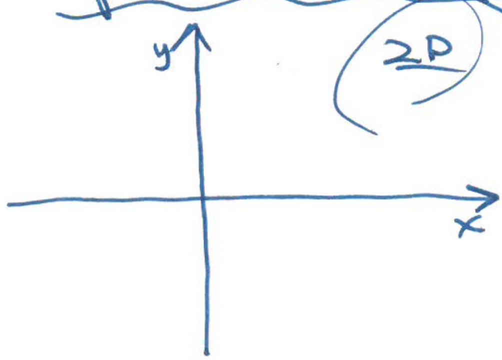
1:00 - 4:00 pm   SAS 2203

80%	$\longrightarrow$	1 pt bonus
85%	$\longrightarrow$	2 pt bonus
90%	$\longrightarrow$	3 pt bonus
95%	$\longrightarrow$	4 pt bonus
100%	$\longrightarrow$	5 pt bonus

6.1: functions of more than one variable

up to today:

2D



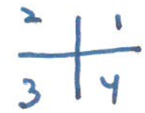
$f(x) = x^2 - 3x + 8$

$f(3) = 3^2 - 3(3) + 8 = \_\_$

ordered pair

$(3, f(3))$

for today:



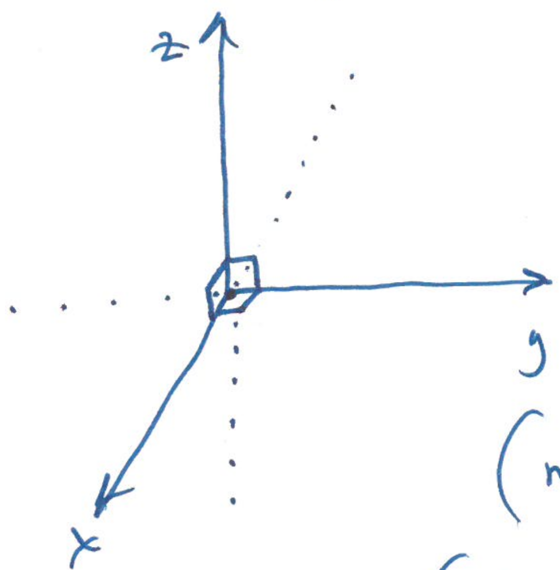
$$f(\underline{x}, \underline{y}) = x^2 - 3xy + y^2 + 2x - y + 1$$

$$f(2, 1) = (2)^2 - 3(2)(1) + (1) + 2(2) - (1) + 1$$

$$= \underline{\qquad\qquad}$$

(x, y, z)

ordered triples



$$f(2, 1) = \underline{\qquad\qquad}$$

$$\left(2, 1, \underset{\substack{\uparrow \\ f(2,1)}}{\underline{\#}}\right)$$

(mutually perpendicular)

8 octants

$(2, 3, 5)$

$(2, 3, -5)$

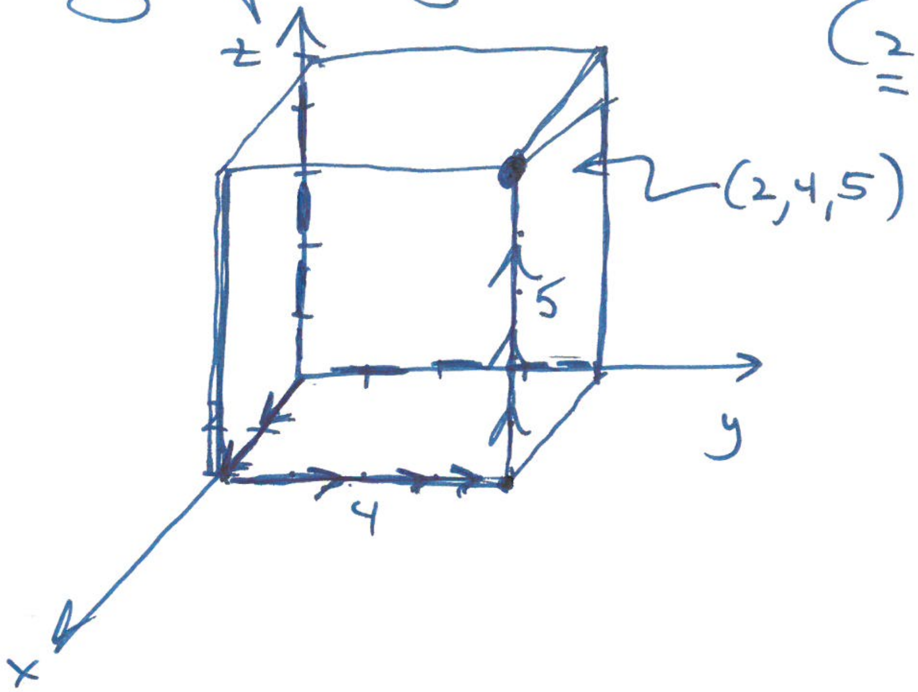
$(2, -3, -5)$

$(2, -3, 5)$

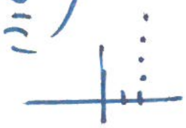
$\vdots$

$\underline{\underline{8}}$

3D graphing:



$(\underline{\underline{2}}, \underline{\underline{4}}, \underline{\underline{5}})$ ③



common example of multi-variate function:

$y = x^2$

$x = y^2$

(T, v, wc)

<u>wind-chill:</u>

$$f(T,v) = \underline{35.74} + 0.\underline{6215}\,T^{1} - 35.75\,V^{.16} + .4275\,(\underline{T}^{1})\,V^{.16}$$

$\underline{v}$ = wind speed (mph)

T = temp. (F°)

$T = \underline{32}^\circ \quad v = 10$ mph

$f(32, 10) = 23.72^\circ$

$(32, 10, \underline{\underline{23.72}})$

6.2: partial derivatives

before ch 6:

$$f(x) = 6x^2 - 5x + 8$$

$$f'(x) = \frac{dy}{dx} =$$

(x, y)

$(x, f(x))$

now:

$$\boxed{f(x, y) = x^2 - 6xy + y^2 = z}$$

(x, y, z) or

$(x, y, x^2 - 6xy + y^2)$

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x$$

partial derivative of the function with respect to x

y is FIXED (x's are VARIABLES)

temp. treat y as if it were CONSTANT.

1st order partial deriv:

$$f(x, y) = x^2 - 6xy + y^2$$

$$f_x = 2x - 6y \cdot (1) + 0 = 2x - 6y$$

$$\boxed{f_x = 2x - 6y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_y = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

y's are variables

(x's are fixed)

(treat x as if it were a constant)

$$f(x,y) = x^2 - 6xy + y^2$$

$$f_y = 0 - 6x(1) + 2y = \boxed{-6x + 2y = f_y}$$

$$\boxed{f_x = 2x - 6y}$$

1st order partial derivatives:

f_x ; f_y

1st order partial deriv:

$f_x \; ; \; f_y$

2nd order partial deriv:

$f_{xx} \; ; \; \boxed{f_{xy}} \; ; \; f_{yy} \; ; \; \boxed{f_{yx}}$

$=$

$$f(x,y) = x^2 \cdot y^3 + 4xy + 4y^2$$

$$f_x = y^3 \cdot (2x) + 4y(1) + 0$$

(x's ARE VAR.)
(y's ARE CONSTANTS)

$$f_y = x^2 \cdot (3y^2) + 4x(1) + 4(2y)$$

(y's ARE VAR.)
(x's ARE CONSTANTS

$$f_x = 2xy^3 + 4y$$

$$f_y = 3x^2y^2 + 4x + 8y$$

2nd order partial deriv:

$f_x = 2xy^3 + 4y \qquad f_x = 2xy^3 + 4y$

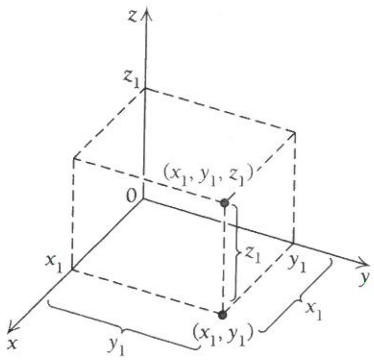
① $f_{xx} = 2y^3(1) + 0 \Rightarrow f_{xx} = 2y^3$

② $f_{xy} = 2x(3y^2) + 4 \Rightarrow f_{xy} = 6xy^2 + 4$

$f_y = \underbrace{3x^2 y^2} + \underbrace{4x} + \underbrace{8y}$ $\qquad$ $f_y = \underbrace{3x^2 y^2} + \underbrace{4x} + \underbrace{8y}$

① $f_{yy} = 3x^2(2y) + 0 + 8 = 6x^2y + 8$

② $f_{yx} = 3y^2(2x) + 4 + 0 = 6xy^2 + 4 = f_{yx}$

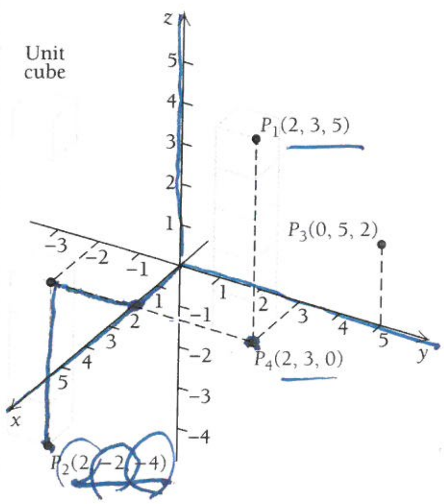


Geometric Interpretations

Visually, a function of two variables, $z = f(x, y)$, can be thought of as matching a point (x_1, y_1) in the xy-plane with the number z_1 on a number line. Thus, to graph a function of two variables, we need a three-dimensional coordinate system. The axes are generally placed as shown to the left. The line z, called the z-axis, is perpendicular to the xy-plane at the origin.

To help visualize this, think of looking into the corner of a room, where the floor is the xy-plane and the z-axis is the intersection of the two walls. To plot a point (x_1, y_1, z_1), we locate the point (x_1, y_1) in the xy-plane and move up or down in space according to the value of z_1.

Unit cube

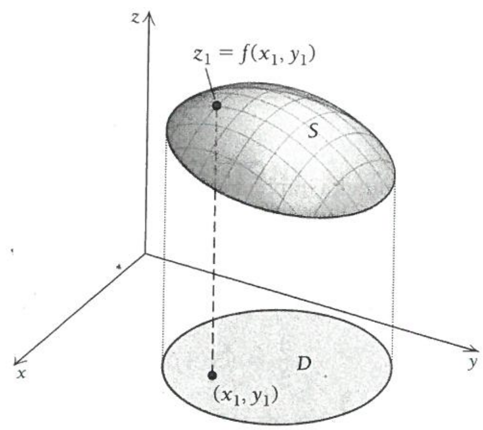


EXAMPLE 7 Plot these points:

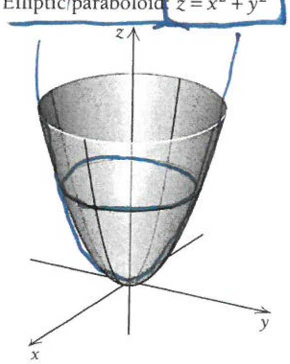
$P_1(2, 3, 5)$,
$P_2(2, -2, -4)$,
$P_3(0, 5, 2)$,
and $P_4(2, 3, 0)$.

Solution The solution is shown at the left. ■

The *graph* of a function of two variables, $z = f(x, y)$, consists of ordered triples (x_1, y_1, z_1), where $z_1 = f(x_1, y_1)$. This graph takes the form of a **surface**. The **domain** of such a function is the set of all points in the xy-plane for which f is defined.



Elliptic paraboloid $z = x^2 + y^2$



EXAMPLE 8 Find the domain of each two-variable function.

a) $f(x, y) = x^2 + y^2$
b) $g(x, y) = \sqrt{1 - x^2 - y^2}$
c) $h(x, y) = x^2 + y^2 + \dfrac{1}{x^2 + y^2}$

Solution

a) Since we can square any real number and add any two squares, f is defined for all x and all y. Therefore, the domain of f is

$$D = \{(x, y) \mid -\infty < x < \infty, \quad -\infty < y < \infty\}.$$

The graph of f is a surface called an *elliptic paraboloid*. A satellite dish is an example of an elliptic paraboloid: the weak incoming signals bounce off the interior surface of the paraboloid and collect at a single point, called the *focus*, thus amplifying the signal.

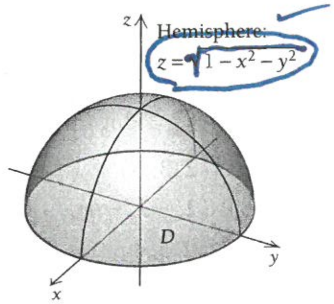
b) For $g(x, y)$ to exist, we must have $1 - x^2 - y^2 \geq 0$, or $x^2 + y^2 \leq 1$. The domain of g is

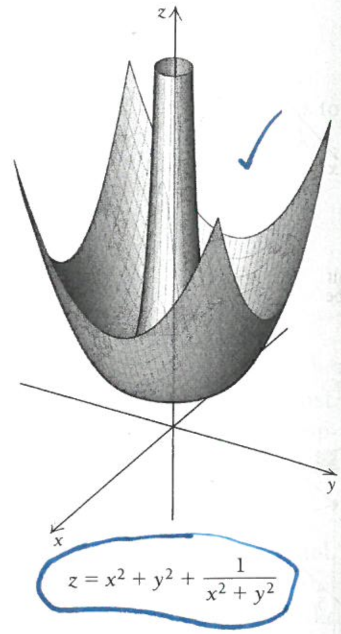
$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

The graph of g is a surface called a *hemisphere,* of radius 1. Its domain is a filled-in circle of radius 1 on the xy-plane. We can think of the domain of g as the "shadow" it casts on the xy-plane.

$z^2 = 1 - x^2 - y^2$

$x^2 + y^2 + z^2 = 1$ ✓





Quick Check 6 ☑

Find the domain of each two-variable function.

a) $f(x, y) = \dfrac{x + y}{x - y}$

b) $g(x, y) = \dfrac{1}{x - 2} + \dfrac{2}{3 + y}$

c) $h(x, y) = \ln(y - x^3)$

c) Since zero cannot be in the denominator, we must have $x^2 + y^2 \neq 0$. Therefore, x and y cannot be 0 simultaneously. The domain of h is

$$D = \{(x, y) \mid (x, y) \neq (0, 0)\}.$$

The graph of h is shown at right. **6** ☑

TECHNOLOGY CONNECTION

Exploratory

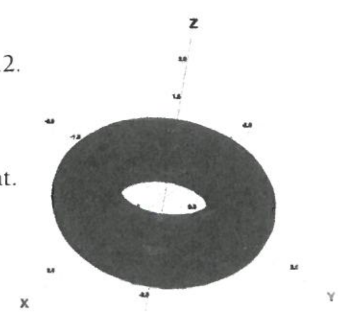
A useful and inexpensive app is Quick Graph, a graphing calculator that creates visually appealing 3D graphs of functions of two variables. It has full graphing interactivity, with touch-based zoom and scroll features.

Some functions and their graphs are presented here.

EXAMPLE 1 Graph:
$(1 - \sqrt{x^2 + y^2})^2 + z^2 = 0.2.$
This is entered as follows:

(1-sqrt(x^2+y^2))^2+z^2=0.2

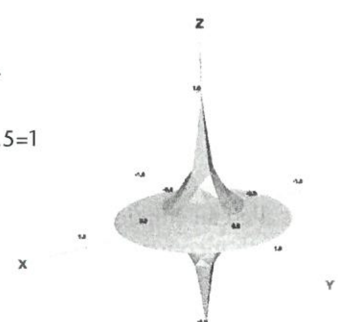
The graph is shown at the right.



EXAMPLE 2 Graph:
$|(2x^2 + 2y^2)^{0.25}| + \sqrt{|z|} = 1.$
This is entered as follows:

abs((2x^2+2y^2)^0.25)+(abs(z))^0.5=1

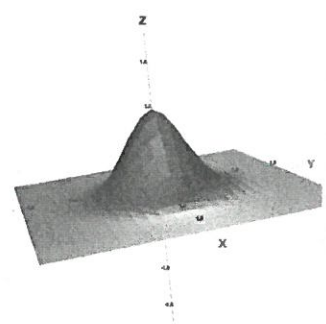
The graph is shown at the right.



EXAMPLE 3 Graph: $z = e^{-4(x^2+y^2)}$.
This is entered as follows:

z=e^(-4(x^2+y^2))

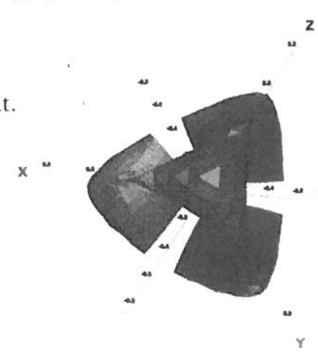
The graph is shown at the right.



EXAMPLE 4 Graph: $(xy)^2 + (yz)^2 + (zx)^2 = xyz.$
This is entered as follows:

(xy)^2+(yz)^2+(xz)^2=xyz

The graph is shown at the right.



(continued)

6.2

- Find the partial derivatives of a given function.
- Evaluate partial derivatives.
- Find the four second-order partial derivatives of a function in two variables.

Teaching Tip

You may need to remind students at times to treat variables as constants. For example, the derivative of $4y^2$, with respect to x, is zero because the derivative of any constant is zero.

Partial Derivatives

Finding Partial Derivatives

Consider the function f given by

$$z = f(x, y) = x^2y^3 + xy + 4y^2.$$

Suppose we fix y at 3. Then

$$f(x, 3) = x^2(3^3) + x(3) + 4(3^2) = 27x^2 + 3x + 36.$$

Note that we now have a function of only one variable. Taking the first derivative with respect to x, we have

$$54x + 3.$$

In general, without replacing y with a specific number, we can consider y fixed. Then f becomes a function of x alone, and we can calculate its derivative with respect to x. This is called the *partial derivative of f with respect to x*, denoted by

$$\frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial z}{\partial x}.$$

Now, let's again consider the function

$$z = f(x, y) = x^2y^3 + xy + 4y^2.$$

The color blue indicates the variable x when we fix y and treat it as a constant. The expressions y^3, y, and y^2 are then also treated as constants. We have

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2y^3 + xy + 4y^2)\\ &= 2xy^3 + (1)y + 0\\ &= 2xy^3 + y.\end{aligned}$$

Similarly, we find $\partial f/\partial y$ or $\partial z/\partial y$ by fixing x (treating it as a constant) and calculating the derivative with respect to y. From

$$z = f(x, y) = x^2y^3 + xy + 4y^2, \qquad \text{The color blue indicates the variable.}$$

we get

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2y^3 + xy + 4y^2)\\ &= x^2(3y^2) + x(1) + 8y\\ &= 3x^2y^2 + x + 8y.\end{aligned}$$

A definition of partial derivatives is as follows.

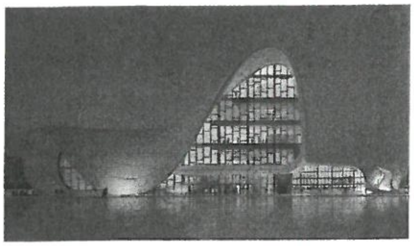
DEFINITION

For $z = f(x, y)$, the **partial derivatives with respect to x and y** are

$$\frac{\partial z}{\partial x} = \lim_{h\to 0}\frac{f(x+h, y) - f(x, y)}{h} \quad \text{and} \quad \frac{\partial z}{\partial y} = \lim_{h\to 0}\frac{f(x, y+h) - f(x, y)}{h}.$$

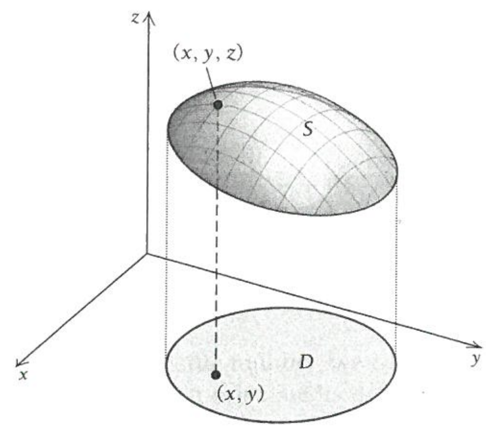
We can find partial derivatives of functions of any number of variables. Since the earlier theorems for finding derivatives apply, we rarely need to use the definition to find a partial derivative.

The Geometric Interpretation of Partial Derivatives

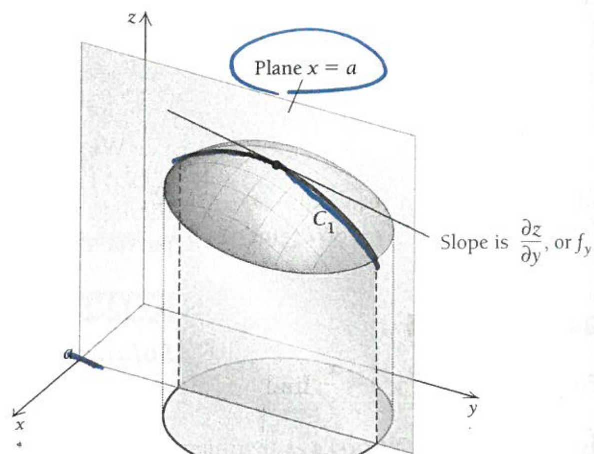


The roof of this building is a smooth continuous surface. The slope at a point on the surface depends on the direction in which the tangent line is oriented.

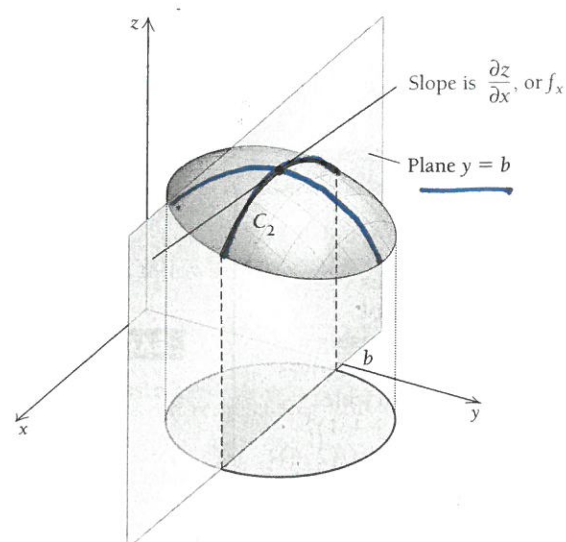
The graph of a function of two variables $z = f(x, y)$ is a surface S, which might have a graph similar to the one shown to the right, where each input pair (x, y) in the domain D has only one output, $z = f(x, y)$.



Now suppose we hold x fixed at the value a. The set of all points for which $x = a$ is a plane parallel to the yz-plane; thus, when x is fixed at a, y and z vary along that plane, as shown to the right. The plane $x = a$ in the figure cuts the surface along the curve C_1. The partial derivative f_y gives the slope of tangent lines to this curve, in the positive y-direction.



Similarly, if we hold y fixed at the value b, we obtain a curve C_2, as shown to the right. The partial derivative f_x gives the slope of tangent lines to this curve, in the positive x-direction.



Higher-Order Partial Derivatives

Consider

$$z = f(x, y) = 3xy^2 + 2xy + x^2.$$

Then $\displaystyle \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = 3y^2 + 2y + 2x.$

Suppose we continue and find the first partial derivative of $\partial z/\partial x$ with respect to y. This will be a **second-order partial derivative** of the original function z, denoted by

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3y^2 + 2y + 2x) = 6y + 2.$$

The notation $\displaystyle \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$ is often expressed as

$$\frac{\partial^2 z}{\partial y\,\partial x} \quad \text{or} \quad \frac{\partial^2 f}{\partial y\,\partial x}.$$

We can also denote the preceding partial derivative using the notation f_{xy}:

$$f_{xy} = 6y + 2.$$

Teaching Tip

You might ask students how many second-order partial derivatives are possible for functions with two variables before discussing the four results and their notation.

Note that in the notation f_{xy}, x and y are in the order (left to right) in which the differentiation is done, but in

$$\frac{\partial^2 f}{\partial y\,\partial x},$$

the order of x and y is reversed. In each case, the differentiation with respect to x is done first, followed by differentiation with respect to y.

Notation for the four second-order partial derivatives is as follows.

DEFINITION Second-Order Partial Derivatives

1. $\displaystyle \frac{\partial^2 z}{\partial x\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial x} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}$ — Take the partial derivative with respect to x, and then with respect to x again.
2. $\displaystyle \frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial y\,\partial x} = f_{xy}$ — Take the partial derivative with respect to x, and then with respect to y.
3. $\displaystyle \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial x\,\partial y} = f_{yx}$ — Take the partial derivative with respect to y, and then with respect to x.
4. $\displaystyle \frac{\partial^2 z}{\partial y\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial y} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$ — Take the partial derivative with respect to y, and then with respect to y again.

EXAMPLE 5 For

$$z = f(x, y) = x^2y^3 + x^4y + xe^y,$$

find the four second-order partial derivatives.

Solution

a) $\displaystyle \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}(2xy^3 + 4x^3y + e^y)$ — Differentiate f with respect to x.

$\displaystyle \qquad = 2y^3 + 12x^2y$ — Differentiate f_x with respect to x.