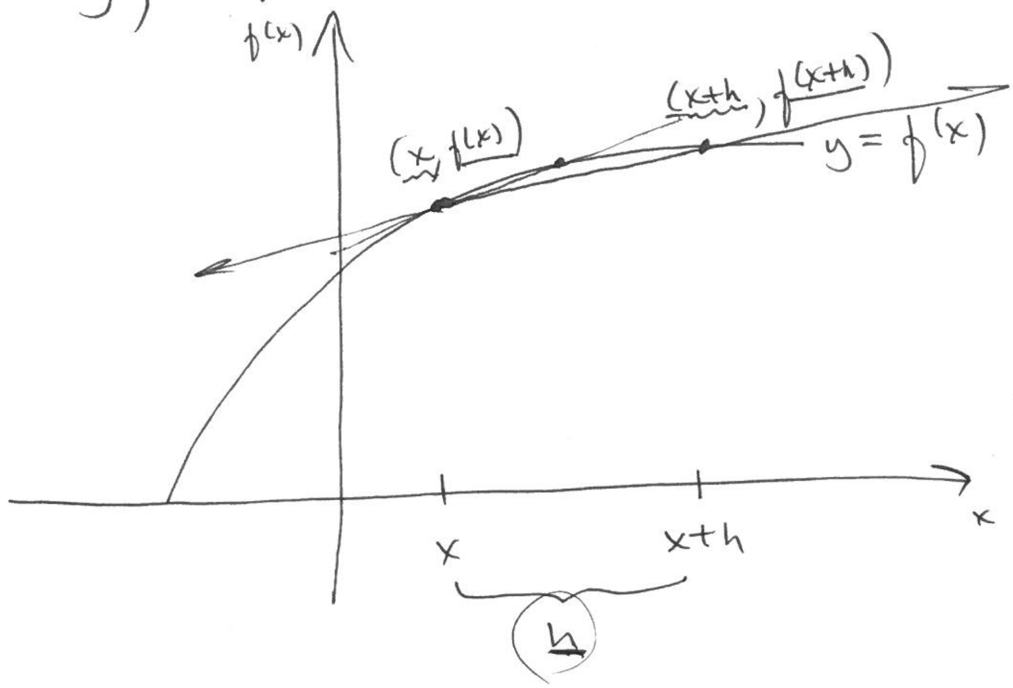
MA 141 - 012

Monday, September 10



x = initial x-value

h = change in x

h = .00000000001

① AVERAGE RATE OF CHANGE

(SLOPE OF THE SECANT LINE) ← 2 pts

(DIFFERENCE QUOTIENT)

$$m_{SEC} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

② INSTANTANEOUS RATE OF CHANGE

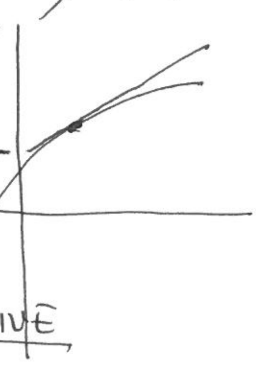
(SLOPE OF THE TANGENT)

(DERIVATIVE)

$\cancel{\dfrac{f(x) - f(x)}{x - x}}$

"prime"

$$m_{TAN} = \lim_{\substack{h \to 0 \\ (h \neq 0)}} \left[\frac{f(x+h) - f(x)}{h} \right] = f'(x)$$



DERIVATIVE

DEFINITION OF DERIV

LINEAR EXAMPLE:

$f(x) = 3x - 1$

$x \to 5$

$$\lim_{x \to 5} f(x) = 14$$

ϵ-δ "proof": $\delta = ??$

if $|x - x_0| < \delta$, then $|f(x) - L| < \epsilon$

before the proof:

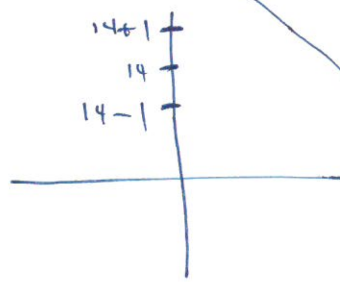
$|(3x-1) - 14| < \epsilon$

"within ϵ units of the limit"

$|3x - 15| < \epsilon$

$3|x-5| < \epsilon$

$|x - 5| < \epsilon/3$



$\epsilon = 1$

$\delta = \epsilon/3 = 1/3$

$\epsilon = \frac{1}{10}$

$\delta = \frac{\epsilon}{3} = \frac{1}{30}$

choose $\delta \le \epsilon/3$

proof:

$$\lim_{x \to 5}(3x-1) = 14$$

(choose $\delta = \epsilon/3$)

if $|x-5| < \delta$

$|x-5| < \epsilon/3$

$3|x-5| < (\epsilon/3) \cdot 3$

$|3(x-5)| < \epsilon$

$|3x - 15| < \epsilon$

$|(3x-1) - 14| < \epsilon \longrightarrow |f(x) - L| < \epsilon$

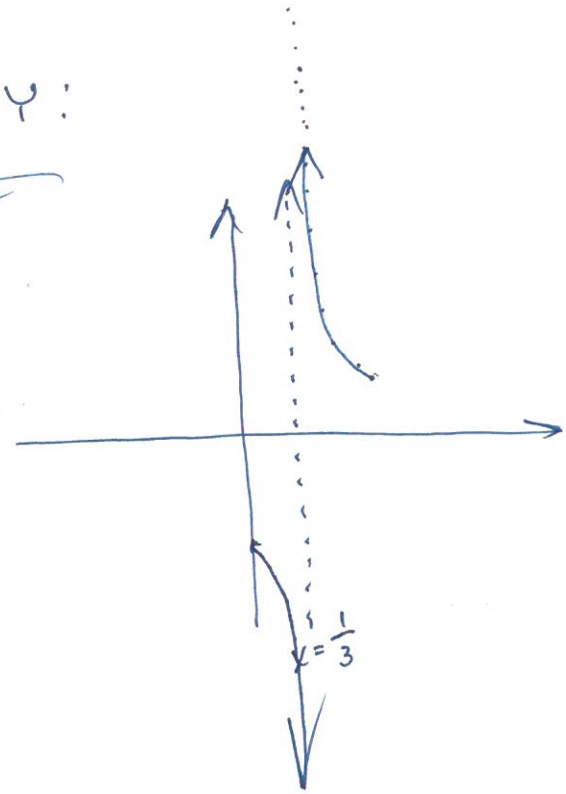
limits at INFINITY:

$$\lim_{x \to \frac{1}{3}^{+}} \frac{4}{3x-1} = \text{D.N.E.} \quad (+\infty)$$

$x \neq \frac{1}{3}$

$x=0 \quad y=-4$

$x=1 \quad y=2$



$$\lim_{x \to \infty} \frac{2x+1}{5x-3} = \frac{2}{5}$$

$$f(x) = \frac{2x+1}{5x-3}$$

H.A.: $y = 2/5$

$$\lim_{x \to \infty} \frac{\frac{2x}{x} + \frac{1}{x}}{\frac{5x}{x} - \frac{3}{x}} = \lim_{x \to \infty} \frac{2 + \overset{0}{\cancel{\frac{1}{x}}}}{5 - \underset{0}{\cancel{\frac{3}{x}}}} = \frac{2}{5}$$

1.1 Introduction

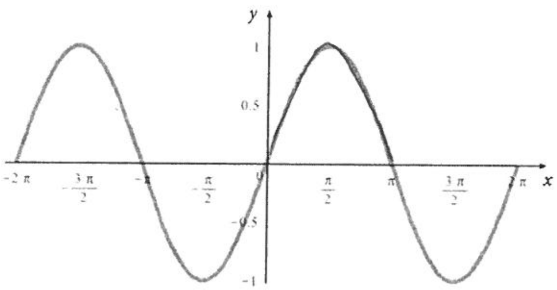


Figure 1

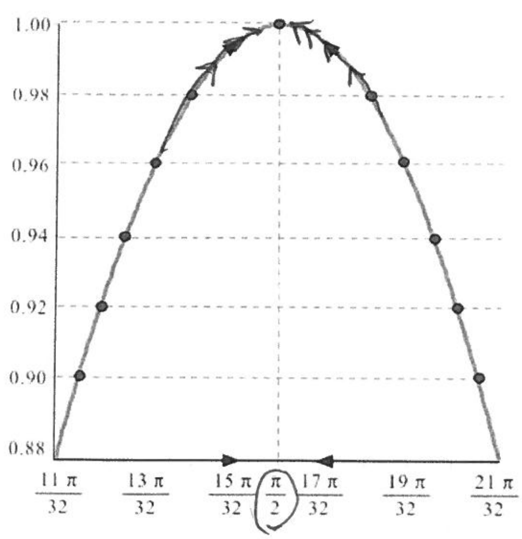


Figure 2

1.1.1 The Limit and Continuous Functions

The graph in Figure 1 depicts two cycles of the function $f(x) = \sin x$. This function is characteristic of the set of "smooth" functions [1], namely those whose graphs vary smoothly over the domain of the function with no breaks or sharp corners. The former quality, that the graph of a "smooth" function like $f(x) = \sin x$ can be sketched without breaks, is referred to as **continuity** on its domain. A more precise definition will have to wait until we have in hand the definition of the limit.

Figure 2 zooms into the region of the plot with

$$\frac{11\pi}{32} \le x \le \frac{21\pi}{32}$$

We notice in the figure that as x approaches $\frac{\pi}{2}$ from either the left or the right, the values of $\sin(x)$ "smoothly" approach the value $\sin(\frac{\pi}{2}) = 1$. This is the behavior we expect of the basic continuous functions such as polynomials, the sine and cosine functions and the exponential functions.

Observation 1. *If f is defined on an open interval I and is continuous at $x_0 \in I$, then as the variable x approaches x_0 from both sides, the values of $f(x)$ approach the value $f(x_0)$. We will use the following symbolic notation to indicate this limiting process:*

$$\lim_{x \to x_0} f(x) = f(x_0) \tag{1}$$

Referring back to Figure 2, we may apply our geometric intuition to conclude that $f(x) = \sin x$ is continuous at $x_0 = \frac{\pi}{2}$, and write

$$\lim_{x \to \frac{\pi}{2}} \sin(x) = \sin\left(\frac{\pi}{2}\right) = 1$$

[1] In Chapter 2 we will give a precise definition of "smooth" functions.

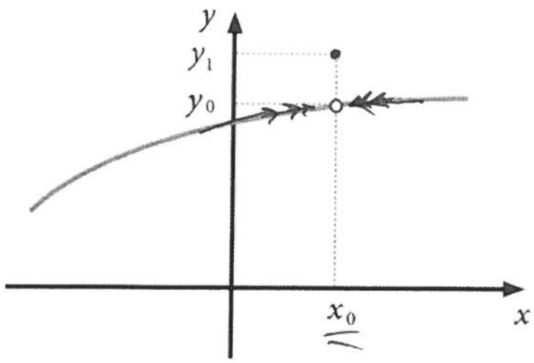


Figure 3

$\lim_{x \to x_0} f(x) = y_0$

However this very nice behavior of the basic functions does not persist for more general functions. In Figure 3 we have the plot of a function with a **removable discontinuity**, defined by

$$f(x) = \begin{cases} \text{left red curve in Figure 3} & \text{if } x < x_0 \\ y_1 & \text{if } x = x_0 \\ \text{right red curve in Figure 3} & \text{if } x > x_0 \end{cases} \tag{2}$$

The key observation in this case is that as the independent variable x approaches the fixed point x_0 the values of the function approach the value y_0 while $f(x_0) = y_1 \neq y_0$. The lesson is:

Observation 2. *A function f can*

1. *be defined at x_0, and*
2. *approach a limiting value y_0 as x approaches x_0 from both sides,*

but y_0 need not be equal to $f(x_0)$.

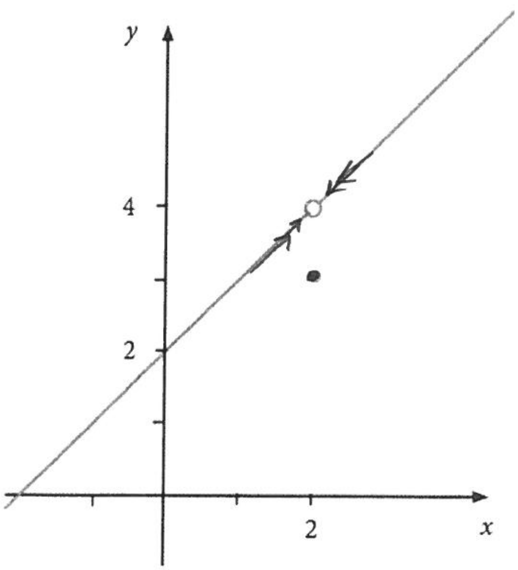


Figure 4

Example 1. *Sketch the graph of the function given by* if $x=2$, $y=4$

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases} \tag{3}$$

$\longrightarrow f(x) = x+2$ $(2,4)$ DELETED

Solution: We saw the first part of this function in Chapter 0. The graph of f is a line with a hole in it. The second part of Formula 3 implies that the domain of this function is $\mathbb{R}$. But the line still has a hole, so f is still discontinuous at $x = 2$. Figure 4 presents a sketch of this function.

If we change the definition of f so that $f(2) = 4$, then the hole is filled in and the discontinuity is removed.

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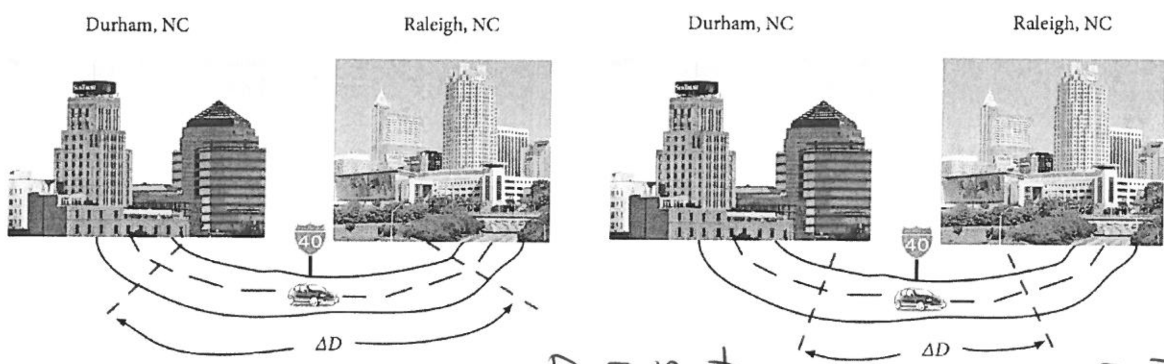
The question to address here is:

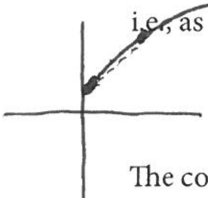
How is the engineer's instantaneous speed at the Interstate 40 marker related to the average speed for the total trip?

A glance at the table shows that in fact there is little correlation. This fact is due to the wide variation in the engineer's speed. At the beginning of the trip the speed is slow as the engineer works his way through the city to I-40. He then accelerates up to highway speed, which varies over the trip, and then his speed slows as he enters Raleigh city limits. Hence instantaneous speed and average speed have little correlation when the distance of the trip is large. To improve the correlation we need to cut down the distance traveled on either side of the central marker so that the variation in speed is reduced. We could cut the distance in half and then to a fourth of the original distance. Comparing the new average speeds over the shortened distances with the instantaneous speed will show a much closer correlation. Continuing to decrease the distance will force the instantaneous speed to be more closely related to the average speed. In fact, what we need is a way to make the distance as short as possible, and this leads us to the idea of *the limit of the average speed as the distance and time for the trip shrink to zero*. The next two illustrations show how we can shrink the distance ΔD of the trip that is centered on the I-40 marker.



$D = r \cdot t$ $\qquad$ $r = \frac{D}{t}$

If we let Δt denote the total time for the middle part of the trip centered on the I-40 marker and denote by ΔD the distance of that portion of the trip, then the distance traveled ΔD is a function of the time traveled Δt and as $\Delta t \to 0$ we have $\Delta D \to 0$. Our definition of instantaneous speed at the I-40 marker is the limit of the average speed as the time for the trip decreases and approaches zero, i.e., as $\Delta t \to 0$. Symbolically we write:



$$\text{Instantaneous Speed} = \lim_{\Delta t \to 0} \text{Average Speed} = \lim_{\Delta t \to 0} \frac{\Delta D}{\Delta t} \tag{1}$$

mi. / hr. $= M_{TAN}$

The concept is intuitive, but the question remains as to how we are to compute this limit. We again need a precise definition of the limit of a function.

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We now apply the same idea to develop the concept of a of a tangent line to an arbitrary smooth curve in the plane. In Figure 12 we have an example of a smooth planar curve that is the graph of a function f. The secant lines connecting P_0 and P_k appear to approach the blue line, which we take to be the tangent line at P_0.

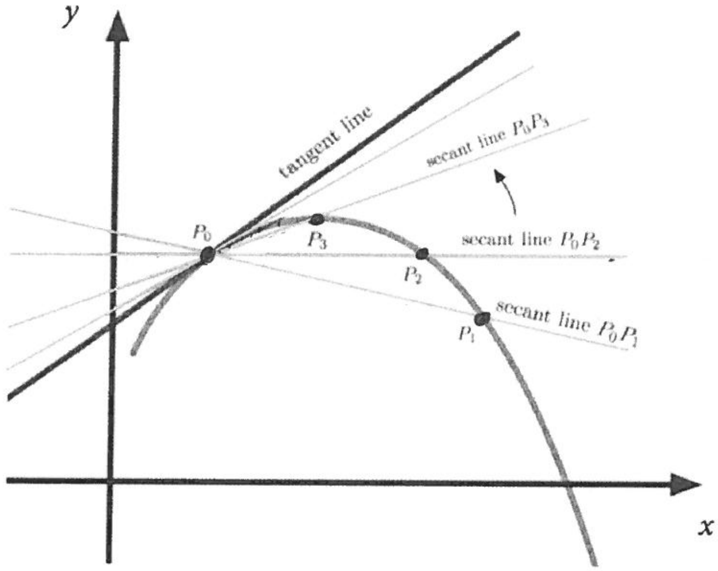
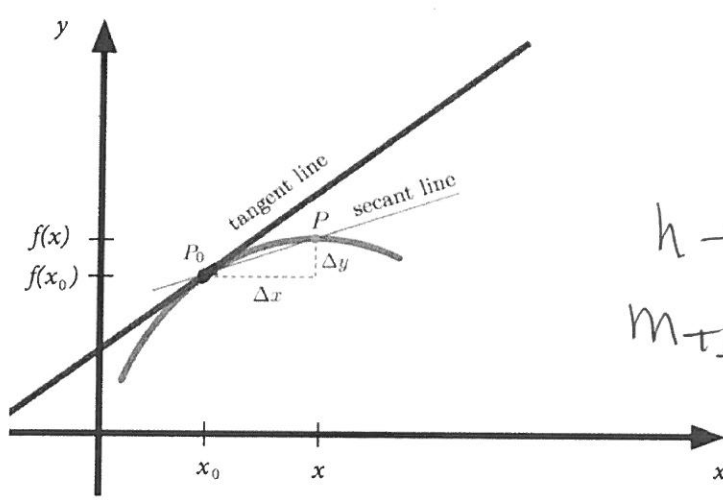


Figure 12: The tangent line at P_0 appears to be approached by the secant lines P_0P_k, $k = 1, 2, 3 \cdots$

To get a better picture and eventually determine the equation of the tangent line, we can zoom into the plot as shown in Figure 13.



$h \to 0$

$m_{TAN} = f'(x)$

Figure 13

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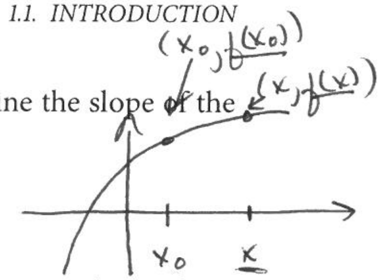

The coordinates $(x_0, f(x_0))$ of P_0 and the coordinates $(x, f(x))$ of P determine the slope of the secant line P_0P, represented by the following difference quotient:

$$m_{\text{secant line}} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}$$

With this notation we see that the point P approaches P_0 as x approaches x_0. Consequently the **slope of the tangent line** will be given by the limit of this difference quotient.

$x \to x_0$

$$\boxed{m_{\text{tangent line}} = \lim_{x \to x_0} m_{\text{secant line}} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} \tag{2}$$

This slope of the tangent line at a point on a curve has many applications. One of these applications is to obtain the equation of the tangent line at a point P_0.

Equation of the Tangent Line

Let the smooth planar curve C be the graph of a function f. The slope of the secant line through $P_0 = (x_0, f(x_0))$ and $P = (x, f(x))$ is given by $\frac{f(x) - f(x_0)}{x - x_0}$. The slope m of the tangent line at P_0 is

$$m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = m_{TAN}$$

So the equation of the tangent line to C at the point $P_0 = (x_0, f(x_0))$ in point-slope form is:

(x_0, y_0)

$$\boxed{y - f(x_0) = \left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right)(x - x_0)} \tag{3}$$

which can be re-written ~~as~~ $y - y_0 = m\ (x - x_0)$

$$y = f(x_0) + \left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right)(x - x_0) \tag{4}$$

Example 2. *Show that $P(1, 2)$ is on the graph of the function $f(x) = 3x^2 - 1$ and find the equation of the tangent line to the graph of f at this point.*

Solution: Since $f(1) = 3 \cdot 1^2 - 1 = 2$, $P(1, 2)$ is on the graph of the function $f(x) = 3x^2 - 1$. The slope of the secant line from $P(1, 2)$ to $(x, f(x))$ (where $x \neq 1$) is

$$m_{\text{secant line}} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(1)}{x - 1} = \frac{3x^2 - 1 - 2}{x - 1} = \frac{3(x+1)(x-1)}{x - 1} = 3(x + 1)$$

The slope of the tangent line to the curve at $P(1, 2)$ is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} 3(x + 1) = 3 \cdot 2 = 6$$

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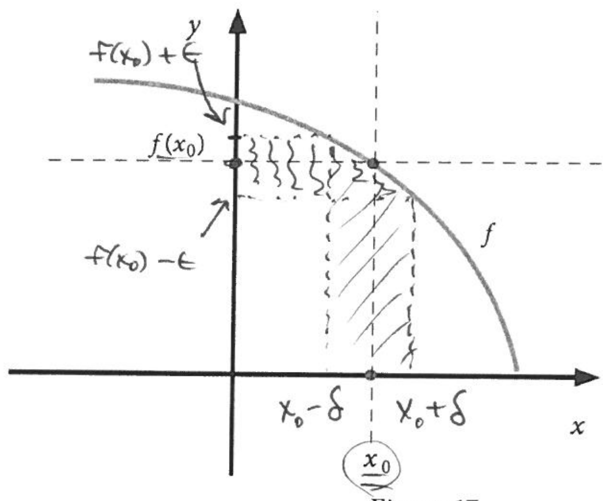
1.2 Definition of the Limit of a Function

In the plot below we have the graph of a smooth function f with the values of $f(x_0)$ and x_0 marked on the y- and x-axes respectively. As we saw in Section 1.1, the values of $f(x)$ approach the value $f(x_0)$ as x approaches x_0 from both sides.



δ = range for "target" x-value

ε = range for "target" y-value

Figure 17

The goal here is to write down a definition of limit that captures this property. The idea goes as follows. Suppose we want to require that the values of $f(x)$ differ in absolute value from $f(x_0)$ by some small positive constant, traditionally called ε. That is, we require $f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$ as illustrated in Figure 18 for a specific choice of ε. This is usually written in the compact form

$$|f(x) - f(x_0)| < \varepsilon \tag{5}$$

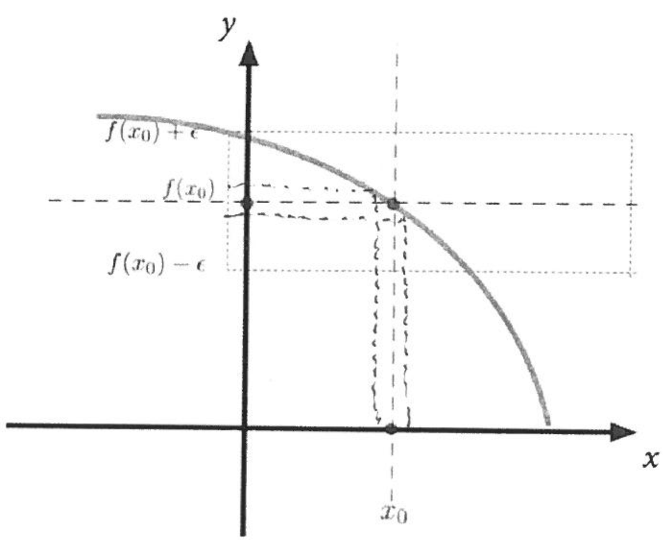
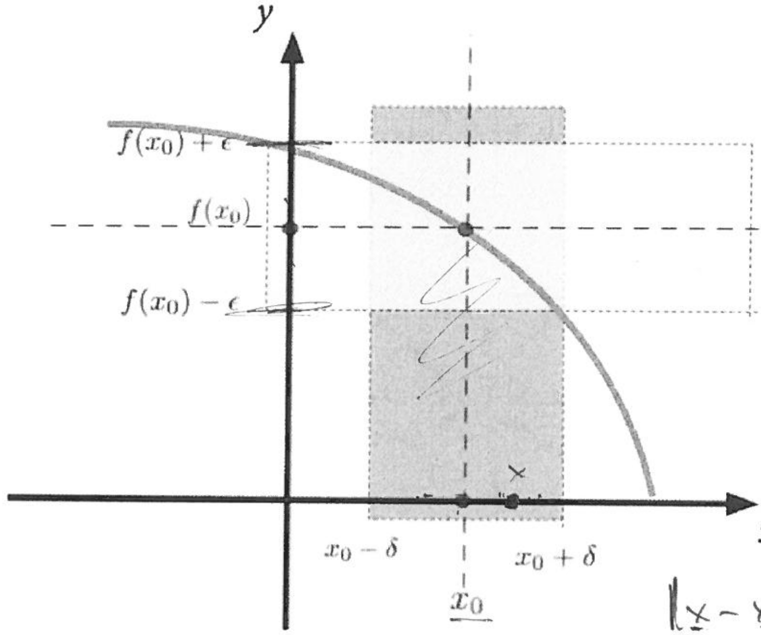


Figure 18

We readily see in this case that Inequality (5) is true for values of x lying inside the vertical blue box in Figure 19, that is, when the values of x satisfy the inequalities $x_0 - \delta < x < x_0 + \delta$, usually written as

$$|x - x_0| < \delta \tag{6}$$



choose "x" in the δ-neighborhood of x_0

$|x - x_0| < \delta$

within δ units of target x-value

Figure 19

You'll notice that this interval is symmetric around x_0 with the choice of δ determined by where the vertical lines at $x_0 - \delta$ and $x_0 + \delta$ intersect the curve. Consequently the interval does not extend as far left as it could; for example, $x = 0 < x_0 - \delta$ clearly satisfies Inequality (5) but is not included in the blue vertical box. Nonetheless we see from Figure 19 that *whenever x is inside the blue vertical box, $f(x)$ is inside the green horizontal box.* Symbolically we write:

Observation 4. *If x satisfies $|x - x_0| < \delta$, then $f(x)$ satisfies $|f(x) - f(x_0)| < \varepsilon$.*

conclusion

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A glance back at Figure 3 in Section 1.1 shows we need a slightly more general concept than that shown in Figure 21 to capture the idea of the limit. Figure 3 shows a situation where certainly the function f has a limit at x_0, but that limit is not $f(x_0)$. Thus we need to replace $f(x_0)$ as the limit point by the symbol L standing for "limit". We will also have to replace $|x - x_0| < \delta$ by the condition $0 < |x - x_0| < \delta$ to allow for this possibility. It is this last step, putting $0 < |x - x_0|$, that truly defines the concept of the limit. While we include points arbitrarily close to x_0, *we do not include x_0 itself.* This allows us to define $L = y_0$ as the limit of f at x_0, despite the fact that $f(x_0) = y_1 \neq y_0$ (in other examples, f might not even be defined at x_0).

The following definition DOES NOT show us how to *compute* a limit, but rather it shows us how to *verify* whether or not a given number L is the limit of a function at a given point. *The power of such an apparently weaker definition is in the theorems we can prove based upon it.* We will develop and study these theorems in the remainder of this chapter.

Definition 1. Limit (2-sided)
Let I be an open interval containing the point x_0. Assume that the function f is defined on a domain $\mathcal{D}$ containing all points of the interval I except possibly x_0. We say that the **limit of the function f at x_0 is L**, *written*

$$\boxed{\lim_{x \to x_0} f(x) = L}$$

hypothesis

if for each $\varepsilon > 0$ there is a number $\delta > 0$ such that $\boxed{if}$ x satisfies $0 < \boxed{|x - x_0| < \delta}$, then $x \in I$ and $f(x)$ satisfies $\boxed{|f(x) - L| < \varepsilon}$.

if $|x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$

REMARK: Informally, this definition says that if we choose x closer and closer to x_0, and if this process forces $f(x)$ closer and closer to L, then

$$\lim_{x \to x_0} f(x) = L$$

In each of the two examples below we take the following approach to applying Definition 1. Given a function f, a point x_0, a candidate L for the limit, and an arbitrary $\varepsilon > 0$, we must find a $\delta > 0$ that works in this particular case. In most cases, the δ we find will depend on ε. Since ε was arbitrary, finding such a δ for this particular ε means that a suitable δ exists for all $\varepsilon > 0$, allowing us to conclude by Definition 1 that $\lim_{x \to x_0} f(x) = L$. We note that the results we obtain using this formal process agree with those we obtained using graphs and geometric intuition in the previous subsections.

Example 3. *Let b be any constant. If $f(x) = b$ for all $x \in \mathbb{R}$, then for all $x_0 \in \mathbb{R}$,*

$$\lim_{x \to x_0} f(x) = b$$

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Solution: We select an arbitrary $\varepsilon > 0$ and look for a $\delta > 0$ such that if x satisfies

$$0 < |x - x_0| < \delta,$$

then $f(x)$ satisfies $|f(x) - b| < \varepsilon$. However since $f(x) = b$,

$$|f(x) - b| = |b - b| = 0 < \varepsilon$$

for all x, so δ is arbitrary and any $\delta > 0$ is acceptable in this extremely trivial case. Thus we conclude that $\lim_{x \to x_0} f(x) = b$. That is, the limit of a constant function $f(x) = b$ is the constant b at every x_0 value.

The following simple result has important implications for computing limits of other functions.

Example 4. *For all $x_0 \in \mathbb{R}$ verify that*

$$\lim_{x \to x_0} x = x_0$$

Solution: Let $f(x) = x$. Selecting an arbitrary $\varepsilon > 0$, we look for a $\delta > 0$ such that whenever x satisfies $0 < |x - x_0| < \delta$, the value of the function f at x satisfies $|f(x) - x_0| < \varepsilon$. Since $f(x) = x$ we have $|f(x) - x_0| = |x - x_0|$. Hence by choosing $\delta = \varepsilon$ we have

$$0 < |x - x_0| < \delta = \varepsilon \quad \Rightarrow 0 < |f(x) - x_0| < \varepsilon$$

Recall that the symbol $\Rightarrow$ means *implies*. Thus the limit of $f(x) = x$ at any $x_0 \in \mathbb{R}$ is x_0.

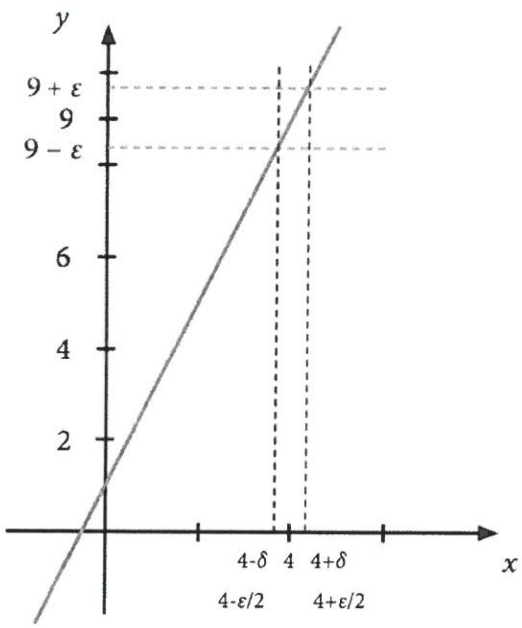

Figure 22

Example 5. *Use Definition 1 to show that*

$$\lim_{x \to 4}(2x + 1) = 9$$

Solution: Let $f(x) = 2x+1$. Selecting an arbitrary $\varepsilon > 0$, we look for a $\delta > 0$ such that whenever x satisfies $0 < |x - 4| < \delta$, the value of the function f at x satisfies $|f(x) - 9| < \varepsilon$. Since $f(x) = 2x + 1$ we have

$$|f(x) - 9| = |(2x + 1) - 9| = |2x - 8| = 2|x - 4|$$

We want $2|x - 4| < \varepsilon$, which is equivalent to $|x - 4| < \frac{\varepsilon}{2}$. By choosing $\delta = \frac{\varepsilon}{2}$ we have

$$0 < |x-4| < \delta = \frac{\varepsilon}{2} \quad \Rightarrow |f(x)-9| = 2|x-4| < 2\delta = \varepsilon$$

Hence the limit of $f(x) = 2x + 1$ at $x = 4$ is 9.

$$\lim_{x \to 4}(2x + 1) = 9$$

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1.2.1 Theorems on Limits

We can use the precise definition of the limit provided in Definition 1 to verify limits in simple cases. However for even mildly complicated functions it is difficult and in most cases impossible to use the definition to verify a limit. Moreover, recall that the definition did not tell us how to find the number L. We need to have a *suitable candidate for* L before we can begin to use the definition. Nonetheless, the definition is such that it may be used to prove a number of theorems on limits. We now quote these well-known basic results which are proved in courses on advanced calculus. Proofs of 5 and 6 in Theorem 1 were given in the previous two examples.

Theorem 1. Theorems on Limits
Assume that functions f and g have limits L_1 and L_2 at the point x_0 so that

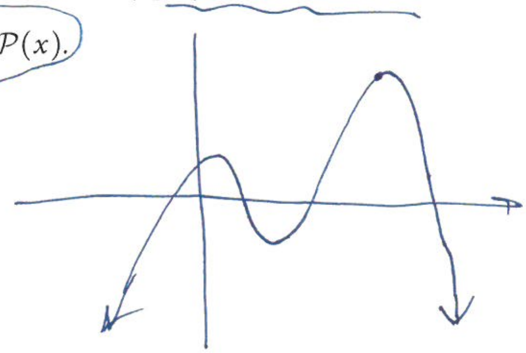
$$\lim_{x\to x_0} f(x) = L_1 \text{ and } \lim_{x\to x_0} g(x) = L_2$$

Then:

1. $\lim_{x\to x_0}(f \pm g)(x) = L_1 \pm L_2$
2. $\lim_{x\to x_0}(fg)(x) = L_1 L_2$
3. $\lim_{x\to x_0}\left(\frac{f}{g}\right)(x) = \frac{L_1}{L_2}$ *provided* $L_2 \neq 0$
4. $\lim_{x\to x_0}(kf)(x) = kL_1$ *for every* k *in* $\mathbb{R}$
5. $\lim_{x\to x_0} b = b$ *for every* b *in* $\mathbb{R}$
6. $\lim_{x\to x_0} x = x_0$

Our next goal is to use Theorem 1 to compute the limit of any polynomial function.

Example 5. *Let* $\mathcal{P}(x) = 7x^2 + 4x + 10$. *Calculate* $\lim_{x\to x_0}\mathcal{P}(x)$.



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Solution: Using the rules from Theorem 1 we find

$$\begin{aligned}
\lim_{x\to x_0} \mathcal{P}(x) &= \lim_{x\to x_0}(7x^2+4x+10) \\
&= \lim_{x\to x_0}(7x^2) + \lim_{x\to x_0}(4x) + \lim_{x\to x_0}(10) \qquad \text{(part 1 of the theorem)} \\
&= 7\lim_{x\to x_0}(x^2) + 4\lim_{x\to x_0}(x) + 10 \qquad \text{(parts 4 and 5 of the theorem)} \\
&= 7\left(\lim_{x\to x_0}(x)\right)^2 + 4x_0 + 10 \qquad \text{(parts 2 and 6 of the theorem)} \\
&= 7x_0^2 + 4x_0 + 10 \qquad \text{(part 6 of the theorem)} \\
&= \mathcal{P}(x_0)
\end{aligned}$$

We anticipated the result of the last example in Section 1.1 for "basic smooth functions", and polynomials fall into this category. It should be clear that we can use the same steps to compute the limit of any given n^{th}-degree polynomial function. However this process would be cumbersome for large n. To prove that this result is true for all polynomials in an efficient way we would need to use the principle of mathematical induction, which we leave to advanced calculus.

Theorem 2. *Let $\mathcal{P}$ be an n^{th} degree polynomial function*

$$\mathcal{P}(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 \ , \qquad n = 0, 1, 2, 3, \ldots \tag{7}$$

NON-NEG INTEGERS

Then for each $x_0 \in \mathbb{R}$

$$\lim_{x\to x_0} \mathcal{P}(x) = \mathcal{P}(x_0) \tag{8}$$

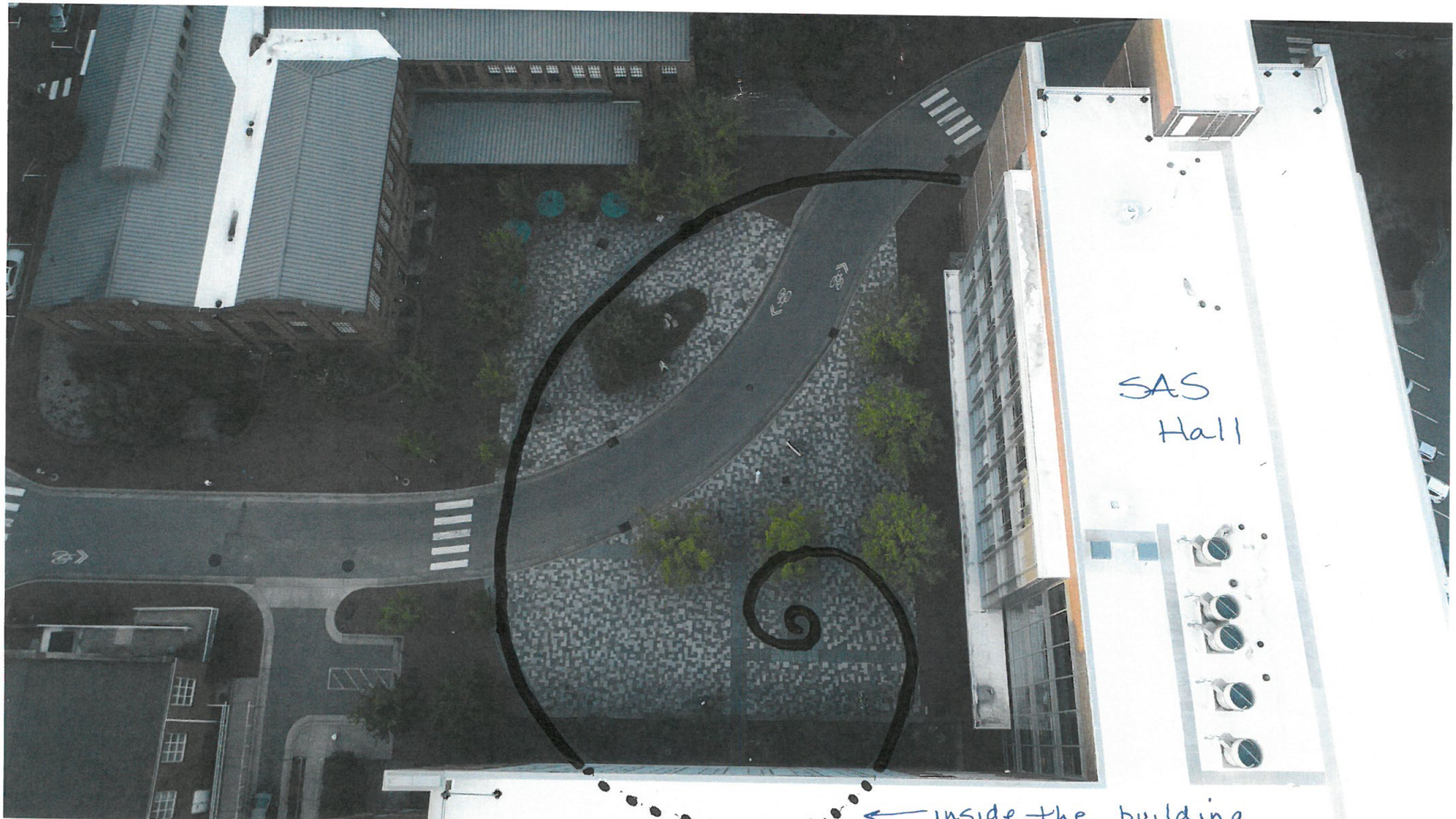
In many situations we will encounter a limit of the form

$$\lim_{x\to x_0} \frac{f(x)g(x)}{g(x)}$$

where $g(x_0) = 0$. This quotient function is not defined at $x = x_0$, but is equal to $f(x)$ at all points where $g(x) \neq 0$. If $g(x)$ is non-zero at the rest of the points in an open interval containing x_0, then this quotient function is equal to $f(x)$ on this open interval except at x_0. Recall, as emphasized in the discussion prior to Definition 1, that the limit of a function at x_0 is determined solely by the values of the function at x-values *near* x_0, and is independent of the behavior of the function *at* x_0. Thus the limit of the quotient function $\dfrac{f(x)g(x)}{g(x)}$ at x_0 is equal to the limit of the simplified function $f(x)$ at x_0 as long as this limit exists.

$$\lim_{x\to x_0} \frac{f(x)g(x)}{g(x)} = \lim_{x\to x_0} f(x)$$

This is a specific case of the following theorem.



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